Math 143 Elementary Algebraic Geometry, Fall 2018. Instructor: Dmitry Tonkonog

Homework Assigment 4, due: September 25

Problem 1. Solve 1.9 in UCA.

Problem 2. Let A be a local ring and x an element in its maximal ideal. Prove that $1+x$ is invertible.

Problem 3. Let A be a ring in which every element satisfies $x^n = x$ (for some $n \geq 2$ depending on x). Show that every prime ideal in A is maximal.

Problem 4. Show that $A = k[x, y]/(x^n, y^m)$ is a local ring, for any $n, m \geq 1$.

Hint: show that any polynomial $f(x, y)$ with non-vanishing constant term is invertible in A. For this, write $f(x, y) = a + f_0(x, y)$ where f_0 has vanishing constant term, write a Taylor expansion for $\frac{1}{a+f_0(x,y)}$ at the origin, and prove that all but finitely many terms in the Taylor expansion are zero in A.

Note: suppose $f_1, \ldots, f_k \in k[x_1, \ldots, x_n]$ are polynomials. Then $k[x_1, \ldots, x_n]/(f_1, \ldots, f_k)$ is a local ring if and only if the solution set $\{f_1 = \ldots = f_k = 0\}$ consists of a single point. In this setting, one can define the intersection multiplicity of the hypersurfaces $\{f_1 = 0\}, \ldots, \{f_k = 0\}$ (intersecting at a single point) purely in terms of the ring $k[x_1, \ldots, x_n]/(f_1, \ldots, f_k)$, in the spirit of how we studied tangent spaces. We may return to this later in the course; you are not required to prove any of this now.