Math 143 Elementary Algebraic Geometry, Fall 2018. Instructor: Dmitry Tonkonog

Homework Assigment 3, due: September 18

Problem 1. Read about quadratic reciprocity and solve 0.13 in UCA.

Problem 2. The ring $\mathbb{Z}[i]$ is sometimes called the ring of Gaussian numbers. For $f = x+iy \in \mathbb{Z}[i]$, define $|f| = x^2 + y^2$. Show that for any $f, g \in \mathbb{Z}[i]$, there exist $q, r \in \mathbb{Z}[i]$ such that

$$f = qg + r$$

and $|r| < |g|$. This means that for Gaussian numbers, one can perform division with remainder. Deduce that every ideal in $\mathbb{Z}[i]$ is generated by one element. See 0.15 in UAC for a hint.

Problem 3. Prove that $3 \in \mathbb{Z}[i]$ is prime, $7 \in \mathbb{Z}[i]$ is prime, but $5 \in \mathbb{Z}[i]$ is not prime. Find the prime decomposition of $5 \in \mathbb{Z}[i]$.

Problem 4. Let p be a prime integer. Prove that p is not prime in $\mathbb{Z}[i] \iff p = z\bar{z}$ where z is a prime Gaussian number $\iff p$ is a sum of two squares (of integers) $\iff$ the equation $x^2 + 1 = 0$ has an integer solution modulo p.

Hint: use that $x^2 + 1 = (x - i)(x + i)$.

Note: arguing a bit more, one can show that a prime integer p is not prime in $\mathbb{Z}[i]$ if and only if $p = 4k + 3$. You are not required to prove this.