Problem 1. Find the singular points of the following affine curves:
(a) $x^2 = x^6 + y^4$,
(b) $xy = x^6 + y^6$.

Problem 2. Let us write down a polynomial $f \in \mathbb{C}[x, y]$ as follows:
\[
f = f_0 + f_1 + f_2 + \ldots
\]
where $f_i \in \mathbb{C}[x, y]$ is homogeneous of degree $i$ (in particular, $f_0 \in \mathbb{C}$ is a constant).

Let $r \geq 0$ be the minimal number such that $f_i \neq 0$. We say that $f$ vanishes to order $r$ at the point $(0, 0)$.

(a) Show that the point $p = (0, 0)$ belongs to the affine curve $\{f = 0\}$ if and only if $f_0 = 0$; and that it is a singular point of the curve if and only if $f_1 = 0$.

(b) Suppose $f$ has degree $d \geq 2$ and vanishes to order $d - 1$ at $(0, 0)$. Let $L \subset \mathbb{C}^2$ be a line passing through $(0, 0)$. Show that $L \cap C$ consists of at most 2 points: the point $(0, 0)$ and another point $p_L$ (which may coincide with $(0, 0)$ for some lines $L$).

(c) In the setting of (b), show that the affine curve $C = \{f(x, y) = 0\}$ admits a rational parametrization. This means that there exist non-constant rational functions $x(t), y(t)$ in one variable $t$ such that
\[
(x(t), y(t)) \in C \text{ for all } t \text{ such that } x(t), y(t) \text{ are defined.}
\]

Recall that a rational function $s$ is a fraction of two polynomials, and for $t \in \mathbb{C}$ one says that $s(t)$ is defined if its denominator does not vanish at $t$.

Hint. Consider a line $L$ given by $x = ty$ where $t$ is a parameter. Let $p_L = (x(t), y(t))$ be the point on $C$ introduced in item (b). Argue that the resulting $x(t), y(t)$ are rational functions. For this, write $f(x, y) = f_{d-1}(x, y) + f_d(x, y)$ where $f_{d-1}$ and $f_d$ are homogeneous of degrees $d - 1$ and $d$, respectively, and solve the following to find $y(t)$:
\[
f_{d-1}(ty, y) + f_d(ty, y) = 0.
\]

Remarks — not part of the homework. (1): projectivizing the statement of item (c), one concludes that there exists a polynomial map $\mathbb{CP}^1 \to \mathbb{CP}^2$ whose image is the projective curve corresponding to $\{f = 0\}$. (2): in contrast, a smooth curve of degree at least 3 does not admit a rational parametrization. We might return to this later in the course.

Problem 3. Solve 1.10 in UAG (Sylvester’s determinant).

Problem 4. Solve 1.2 in UCA (about the product of ideals).