Math 143 Elementary Algebraic Geometry, Fall 2018. Instructor: Dmitry Tonkonog

Homework Assigment 1, due: September 11

Problem 1. Find the singular points of the following affine curves:

(a) $x^2 = x^6 + y^4$,

(b) $xy = x^6 + y^6$.

Problem 2. Let us write down a polynomial $f \in \mathbb{C}[x, y]$ as follows:

$$f = f_0 + f_1 + f_2 + \dots$$

where $f_i \in \mathbb{C}[x, y]$ is homogeneous of degree i (in particular, $f_0 \in \mathbb{C}$ is a constant).

Let $r \geq 0$ be the minimal number such that $f_i \neq 0$. We say that f *vanishes to order* r at the point $(0, 0)$.

(a) Show that the point $p = (0, 0)$ belongs to the affine curve $\{f = 0\}$ if and only if $f_0 = 0$; and that it is a singular point of the curve if and only if $f_1 = 0$.

(b) Suppose f has degree $d \geq 2$ and vanishes to order $d - 1$ at $(0, 0)$. Let $L \subset \mathbb{C}^2$ be a line passing through $(0, 0)$. Show that $L \cap C$ consists of at most 2 points: the point $(0, 0)$ and another point p_L (which may coincide with $(0, 0)$ for some lines L).

(c) In the setting of (b), show that the affine curve $C = \{f(x, y) = 0\}$ *admits a rational parametrization.* This means that there exist non-constant rational functions $x(t), y(t)$ in one variable t such that

$$(x(t), y(t)) \in C \text{ for all } t \text{ such that } x(t), y(t) \text{ are defined.}$$

Recall that a *rational function* s is a fraction of two polynomials, and for $t \in \mathbb{C}$ one says that $s(t)$ is *defined* if its denominator does not vanish at t.

Hint. Consider a line L given by $x = ty$ where t is a parameter. Let $p_L = (x(t), y(t))$ be the point on C introduced in item (b). Argue that the resulting $x(t), y(t)$ are rational functions. For this, write $f(x, y) = f_{d-1}(x, y) + f_d(x, y)$ where f_{d-1} and f_d are homogeneous of degrees $d - 1$ and d, respectively, and solve the following to find $y(t)$:

$$f_{d-1}(ty, y) + f_d(ty, y) = 0.$$

Remarks — not part of the homework. (1): projectivizing the statement of item (c), one concludes that there exists a polynomial map $\mathbb{C}P^1 \to \mathbb{C}P^2$ whose image is the projective curve corresponding to $\{f = 0\}$. (2): in contrast, a smooth curve of degree at least 3 does not admit a rational parametrization. We might return to this later in the course.

Problem 3. Solve 1.10 in UAG (Sylvester's determinant).

Problem 4. Solve 1.2 in UCA (about the product of ideals).