Problem 1. Let $A$ be an Artinian ring which is a domain (it has no zerodivisors). Prove that $A$ is a field.

Problem 2. Let $A$ be Artinian, and $m_1, \ldots, m_n$ its maximal ideals. Verify that $(m_i)^r$ and $(m_j)^r$ are coprime, for all $i \neq j$ and all $r > 0$. This means that you need to check $(m_i)^r + (m_j)^r = 1$. (This was a necessary thing to check in the proof of the structure theorem for Artinian rings, in order to apply the Chinese remainder theorem).

Problem 3. Let $A$ be a local Artinian ring with maximal ideal $m$, and let $k = A/m$. Prove that the following are equivalent:

(1) Every ideal $I \subset A$ is principal (generated by one element);
(2) $m$ is principal;
(3) the dimension of $m/m^2$ as a $k$-vector space is at most 1;
(4) every ideal in $A$ is a power of $m$.

Hint: (1)⇒(2)⇒(3), (1)⇒(4) and (4)⇒(1) should be easy. To prove (1)⇒(3), consider two cases.

If $\dim_k m/m^2 = 0$, argue that $m = 0$ using Nakayama’s lemma. Hence $A$ is a field.

If $\dim_k m/m^2 = 1$, pick an element $x \in m^2 \setminus m$. Show that $m = (x)$ using Nakayama’s lemma. Let $I$ be an ideal, then for some $r$, $I \subset m^r$ and $I \not\subset m^{r+1}$. So there is $y \in I$ such that $y = ax^r$ but $y \notin (x^r+1)$. Prove that $I = (x^r)$. 

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