Math 143 Elementary Algebraic Geometry, Fall 2018. Instructor: Dmitry Tonkonog

Homework Assigment 10, due: **November 13**

Problem 1. Let A be an Artinian ring which is a domain (it has no zerodivisors). Prove that A is a field.

Problem 2. Let A be Artinian, and $m_1, \ldots, m_n$ its maximal ideals. Verify that $(m_i)^r$ and $(m_j)^r$ are coprime, for all $i \neq j$ and all $r > 0$. This means that you need to check $(m_i)^r + (m_j)^r = 1$. (This was a necessary thing to check in the proof of the structure theorem for Artinian rings, in order to apply the Chinese remainder theorem).

Problem 3. Let A be a local Artinian ring with maximal ideal m, and let $k = A/m$. Prove that the following are equivalent:

(1) Every ideal $I \subset A$ is principal (generated by one element);

(2) m is principal;

(3) the dimension of m/m^2 as a k-vector space is at most 1;

(4) every ideal in A is a power of m.

Hint: (1)⇒(2)⇒(3), (1)⇒(4) and (4)⇒(1) should be easy. To prove (1)⇒(3), consider two cases.

If $\dim_k m/m^2 = 0$, argue that $m = 0$ using Nakayama's lemma. Hence A is a field.

If $\dim_k m/m^2 = 1$, pick an element $x \in m^2 \setminus m$. Show that $m = (x)$ using Nakayama's lemma. Let I be an ideal, then for some r, $I \subset m^r$ and $I \not\subset m^{r+1}$. So there is $y \in I$ such that $y = ax^r$ but $y \notin (x^{r+1})$. Prove that $I = (x^r)$.