Math 143 Elementary Algebraic Geometry, Fall 2018. Instructor: Dmitry Tonkonog

Homework Assigment 1, due: September 4

Problem 1. Consider the map

$$\sigma : \mathbb{P}^{r-1} \times \mathbb{P}^{s-1} \to \mathbb{P}^{rs-1}$$

given by the formula

$$([x_1 : x_2 : \ldots : x_r], [y_1 : y_2 : \ldots : y_s]) \mapsto [x_1y_1 : x_1y_2 : \ldots x_1y_s : x_2y_1 : x_2y_2 : \ldots]$$

It is called the Segre embedding.

(a) For convenience, let us index the coordinates on $\mathbb{C}^{rs}$ by the symbols z_{ij} where $i = 1, \ldots, r$ and $j = 1, \ldots, s$. Show that the image of the Segre embedding is given by the collection of equations $\{z_{ij}z_{kl} = z_{il}z_{kj}\}$.

(b) Show that σ is indeed an embedding (is injective).

Hint: consider the special case $r = s = 2$. Then $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ is given by

$$([x_1 : x_2], [y_1 : y_2]) \mapsto [x_1y_1 : x_1y_2 : x_2y_1 : x_2y_2].$$

Let $[u : v : z : w]$ be the coordinates on $\mathbb{P}^3$, then item (a) says that the image of σ is given by the single equation $uw = vz$. We now see that the projective conic in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Problem 2. Solve 1.9 in [UAG]: let $p_1, \ldots, p_4$ be 4 points in $\mathbb{P}^2$ where no 3 points are collinear. Without loss of generality, assume that these points are $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$, $[1 : 1 : 1]$. (Recall that $PGL(3)$ acts transitively on 4-tuples of points in $\mathbb{P}^2$, where no 3 points are collinear).

(a) Find all conics which pass through the 4 points above.

Hint: start with a general conic $ax^2 + by^2 + cz^2 + exy + fxz + gxz$ and see what it means for it to pass through the above points.

(b) Prove that there exists a conic passing through any 5 points (you may assume that no 3 points are collinear).

Hint: find a conic from part (a) which additionally passes through any given fifth point.

Problem 3. Give an example (explicit equations in projective coordinates) of two smooth projective conics in $\mathbb{C}P^2$ that intersect in exactly two distinct points, where one intersection has multiplicity 3 and the other has multiplicity 1. A conic is smooth is and only if it is not a union of two lines and not a double line.

Hint: It is easy to guess an example of two conics $f_1 = 0$, $f_2 = 0$ with f_1 is a union of two lines and f_2 smooth, which have the desired intersection multiplicities. For example, take $f_1(x, y) = xy$ (a union of 2 lines in affine coordinates), and let $f_2 = 0$ be a circle passing through the origin and tangent to the x-axis. Write an explicit equation. Then it intersects the x-axis with multiplicity 2 and the y-axis with multiplicity 1. The total intersection multiplicity of $f_1 = 0$ and $f_2 = 0$ at the origin should hence be 3. This is verified by the Bezout theorem and the fact that $f_1 = f_2 = 0$ has only one more solution except for $x = y = 0$ (why?), and that other intersection point is transverse (why?). See the next page.

Now replace the singular conic $f_1 = 0$ *with the conic* $f_1 + \alpha f_2 = 0$ *for some* $\alpha \in \mathbb{C}$, $\alpha \neq 0$. *Why is it smooth, for generic* α*? What can you say about the intersection of the conics* $f_1 + \alpha f_2 = 0$ *and* $f_2 = 0$*?*