Math 143 Elementary Algebraic Geometry, Fall 2018. Instructor: Dmitry Tonkonog

Practice Final

The final will be 3 hours long. No notes, books or calculators are allowed.

Problem 1. Compute the intersection multiplicity at the origin of the following two affine curves:

$$\begin{cases}(x^2+y^2)^2+3x^2y-y^3=0,\\ (x^2+y^2)^3-4x^2y^2=0.\end{cases}$$

Problem 2. Find all intersection points of the following two curves in $\mathbb{C}P^2$, and find the multiplicity of each point:

$$\begin{cases}y^2z-x^3+xz^2=0,\\ x-z=0.\end{cases}$$

Problem 3. Let $X \subset \mathbb{C}P^2$ be the the union of two lines intersecting at a single point. Compute the Hilbert function $h_X(d)$.

Problem 4. (a) What does it mean for a ring A to be *normal*? Give the definition.

(b) Consider $f = y^2 - x^2 - x^3$ and $A = \mathbb{R}[x,y]/(f)$. Show that A is not normal.

Hint: show that the element $y/x \in \operatorname{Frac} A$ is integral over A.

(c) Prove that the integral closure of the ring A from item (b) is isomorphic to $\mathbb{R}[t]$.

Hint: concretely, the integral closure of A is $\mathbb{R}[\frac{y}{x}] \subset \operatorname{Frac} A$.

Problem 5. Let A be a Noetherian ring, and $I \subset A$ an ideal. Prove that there exists a positive integer r such that $(\operatorname{rad} I)^r \subset I$.

Note: we proved this in class; you can repeat the proof.

Problem 6. (a) What does it mean for a ring A to be *Artinian*? Give the definition.

(b) Suppose $I \subset k[x_1,\ldots,x_n]$ is an ideal such that $V(I) = \{p_1,\ldots,p_s\}$ consists of finitely many points. Suppose that $g \in k[x_1,\ldots,x_n]$ is a polynomial such that

$$g(p_1) = \ldots = g(p_s) = 0.$$

Prove that for some positive integer r, $g^r \in I$.

Problem 7. Let R be a graded Noetherian ring.

(a) What does it mean for an ideal $I \subset R$ to be *homogeneous*? Give the definition.

(b) Prove that $\operatorname{nilrad} R$ is a homogeneous ideal in R.

Hint: suppose $f \in R$ is nilpotent. Write $f = \sum f_i$ as a finite sum of homogeneous elements, $f_i \in R_i$. Show that each f_i is also nilpotent.

To do so, use induction on the number s of non-trivial summands in $f = \sum f_i$. Suppose $f^n = (\sum f_i)^n = 0$, expand this power, and look at the highest degree summand in the expansion.

Problem 8. Let $A = k[x_1, \dots, x_n]$, $f, g \in A$ be coprime (i.e. they are not divisible by a common non-constant polynomial h), and $I = (f, g)$. Prove that the following is a short exact sequence of A-modules:

$$0 \to A \xrightarrow{(-g,f)} A \oplus A \xrightarrow{\binom{f}{g}} I \to 0.$$

Above, the first map takes h to $(-gh, fh)$ and the second map takes (h_1, h_2) to $fh_1 + gh_2$.

See the next page for the solutions.

Problem 1. Answer: 14. Use the algorithm explained in class.

Problem 2. Answer: There are two intersection points, $[1:0:1]$ and $[0:1:0]$. Their intersection multiplicities are 2 and 1, respectively.

Solution: First, one computes the intersection points; I skip this part. Let us, for example, compute the intersection multiplicity of $[0:1:0]$. This point lies in the affine chart $y=1$ (note that it does not lie in the charts $x=1$ or $z=1$, so I have to use the chart $y=1$). Setting $y=1$ in the curve equations, I get the following:

$$z-x^3+xz^2=0,$$
$$x-z=0.$$

The point under consideration is $(0,0)$ in this chart. The intersection multiplicity is 1, and it can be verified in several different ways.

The first way is to compute the derivatives: $(0,1)$ and $(1,-1)$, and notice that they are not proportional to each other, so the intersection is transverse, which is the same as the fact that it has multiplicity 1.

The second way to verify this is to use our algorithm.

$$\mu_0(-x^3+xz^2+z,\ x-z)=\mu_0(-x^3+xz^2+z+x^2(x-z),\ x-z)=\mu_0(z,x-z)=1.$$

The third way is to note that $x-z$ is a line, so the intersection multiplicity if the multiplicity of 0 as a root of the equation $z-x^3+xz^2=0$ after we substitute $x=z$. It becomes:

$$-x^3+x^3+x=x,$$

so the multiplicity is 1.

By Bezout's theorem, the multiplicity at $[1:0:1]$ is 2. Note that you could have computed the multiplicity at $[1:0:1]$ using the algorith, too. Don't forget to pass to an affine chart in order to run the algorithm.

Problem 3. Answer: $h(d)=2d+1$ for $d\geq 1$.

Solution: Let $[x_0:x_1:x_2]$ be the homogeneous coordinates on $\mathbb{C}P^2$. Because the group $PGL(3)$ of projective transformations acts transitively on $\mathbb{C}P^2$, we may assume that the intersection point is $[1:0:0]$. In the affine chart $\mathbb{C}^2$ where the last projective coordinate is set to 1, X become a union of two affine lines intersecting at the origin. There is a linear map $\mathbb{C}^2\to\mathbb{C}^2$ taking that union of affine lines to $\{x_0x_1=0\}$. This map induces a projective transformation $\mathbb{C}P^2\to\mathbb{C}P^2$ taking the whole projective variety X to the one given by $x_0x_1=0$.

Let us compute the dimension of

$$\frac{\mathbb{C}[x_0,x_1,x_2]_d}{(x_0x_1)_d}.$$

The basis of this vector space over $\mathbb{C}$ consists of monomials that are not divisible by x_0x_1, and the list of such monomials is:

$$x_0^d,\ x_0^{d-1}x_2,\ x_0^{d-2}x_2^2,\dots,x_2^d,$$
$$x_1^d,\ x_1^{d-1}x_2,\ x_1^{d-2}x_2^2,\dots,x_1x_2^{d-1}.$$

There are $2d+1$ monomials in this list. So the dimension of
$$\frac{\mathbb{C}[x_0,x_1,x_2]_d}{(x_0x_1)_d}$$
is $2d+1$.

Problem 4. (a) A is normal if A equals its integral closure inside $\mathrm{Frac}(A)$.

(b) Let $t=\frac{y}{x}$, then in $\mathrm{Frac}A$ we have $t^2-x-1=0$, which means that t is in the integral closure of A. Because $t\notin A$, A is not normal.

(c) Let $\bar{A}\subset \mathrm{Frac}A$ be the integral closure of A. We have shown in (b) that $A\subset\mathbb{R}[\frac{y}{x}]\subset\bar{A}$. It was shown in class that $\mathbb{R}[t]$ is a normal ring, so $\overline{\mathbb{R}[\frac{y}{x}]}=\mathbb{R}[\frac{y}{x}]$. Then the above chain of inclusions implies $\bar{A}\subset\overline{\mathbb{R}[\frac{y}{x}]}=\mathbb{R}[\frac{y}{x}]\subset\bar{A}$, which means $\mathbb{R}[\frac{y}{x}]=\bar{A}$.

Problem 5. Suppose $\operatorname{rad} I=(f_1,\dots,f_n)$. There exist $r_i\in\mathbb{Z}_{>0}$ such that a $f_i^{r_i}\in I$. Let $r=\sup\{r_i\}_{i=1,\dots,n}$. Then any element of I has the form $a_1f_1+\ldots+a_nf_n$ for some $a_i\in A$. We claim that $(a_1f_1+\ldots+a_nf_n)^{rn}\in I$. Indeed, by the Binomial Theorem, every term in the expansion of that power has the form $\ldots\cdot f_1^{i_1}\ldots f_n^{i_n}$ with $\sum i_j\geq rn$. So at least one of the coefficients i_j satisfies $i_j\geq r$. Then the corresponding element $f_j^{i_j}\in I$.

Problem 6. (a) A is Artinian if it satisfies the descending chain condition for ideals: every descending chain of ideals
$$I_1\supset I_2\supset\ldots$$
eventually stabilizes: $I_j=I_{j+1}=\ldots$

(b) By theorems proved in class: $A=k[x_1,\dots,x_n]/I$ is Artinian, and it holds in A that $(m_{p_1}\cdot\ldots m_{p_r})^r=0$ for some r. Here m_{p_i} is the maximal ideal consisting of all functions vanishing at p_i. We are given that $g\in m_{p_i}$ for all i, consequently $g\in m_{p_1}\cap\ldots\cap m_{p_s}=m_{p_1}\ldots m_{p_s}$. So $g^r=0$ in A, wich means $g^r\in I$.

Problem 7. Suppose $f^n=0$, write $f=\sum f_i$ as a sum of homogeneous elements of degree i. We show by induction on the number s of summands that all the f_i are nilpotent. When $s=0$, the claim is tautological. Otherwise, let f_m be the highest-degree term, so f_m^n is the unique highest-degree term of f^n, and is therefore zero. Since f_m is nilpotent, so is $f-f_m$, which has $s-1$ summands, all of which are nilpotent by induction.

By the Noetherian property, $I=\operatorname{nilrad}A$ is generated by a finite collection of polynomials. Then it is also generated by the collection of all homogeneous summands of all these polynimials, by what is shown above.

Problem 8. Let us check, for example, that the kernel of the second map lies in the image of the first map (the rest is very easy). Suppose $fh_1+gh_2=0$. Because f,g are coprime, h_1 is divisible by g: $h_1=gH_1$. Similarly, $h_2=fH_2$. We have
$$fgH_1+fgH_2=0,$$
so $H_1=-H_2$. Denote $H_1=h$, then $(h_1,h_2)=(gh,-fh)$, so it is in the image of the first map.