Math 185 Complex Analysis, Fall 2017. Instructor: Dmitry Tonkonog

Practice Final

The final will be 3 hours long, no notes allowed.

1. Compute the real part of

$$\left(\frac{1-i}{1+i}\right)^3.$$

2. Suppose $f(z)$ is an analytic function on a domain D and $\arg f(z)$ is constant on D. Prove that $f(z)$ is constant on D.

3. Consider the function defined by

$$f(z) = \frac{1}{\sin z} - \frac{1}{z}, \quad z \neq 0$$

and

$$f(0) = a$$

for some $a \in \mathbb{C}$. Prove that that there exists a value $a \in \mathbb{C}$ for which $f(z)$ is an analytic function in a neigborhood of $z = 0$, and determine that value of a.

4. Let $\sum_{n=0}^{\infty} a_n z^n$ be the Taylor expansion of

$$f(z) = \frac{1}{z + \cos z}$$

at the origin. Find a_0, a_1, a_2.

5. Suppose that $f(z)$ is analytic in a disk $|z| < \epsilon$ for some $\epsilon > 0$. Assume that $f(z)$ satisfies

$$f(z) = z + f(z^2)$$

for $|z| < \epsilon$ and

$$f(0) = 0.$$

Prove that for all z such that $|z| < \epsilon$,

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}.$$

6. Compute the residue

$$\operatorname*{Res}_{z=0} \frac{1}{z^4(2-z)}.$$

7. Compute the integral

$$\int_C \frac{1}{(z-1)^2\,(z^2+1)} dz$$

where the contour C is the circle $|z - 1 - i| < 2$ oriented counterclockwise.

8. Compute the integral

$$\int_{-\infty}^{+\infty} \frac{\cos ax}{(x^2+b^2)^2}$$

9. Consider the polynomial

$$2z^4 - 5z + 2.$$

Find the number of its roots within the disk $|z| < 1$, counted with multiplicities.

10. The Gauss mean value theorem states the following. Suppose a function $f(z)$ is analytic in the region $|z - a| \le R$, then

$$\frac{1}{2\pi}\int_0^{2\pi} f(a + Re^{i\phi})d\phi = f(a).$$

Prove this theorem using the Cauchy integral formula.

Answers and solutions can be found on the next page.

Answers and solutions.

1. Answer: 0.

2. Proof. Let $f(z) = u(x,y) + iv(x,y)$. The condition that $\arg f(z)$ is constant means that either

$$u(x,y) = cv(x,y)$$

for some constant $c \in \mathbb{R}$, or that

$$v(x,y) = 0.$$

Assume the former. Then

$$u_x = cv_x, \quad u_y = cv_y.$$

By the Cauchy-Riemann equations,

$$u_x = v_y, \quad u_y = -v_x.$$

Combining the above,

$$u_x = cv_x = -cu_y = -c^2 v_y = -c^2 u_x.$$

It follows that $u_x = 0$. Analogously $u_y = 0$, $v_x = 0$, $v_y = 0$. This holds at any point of D, therefore u and v are constant, therefore f is constant.

The case $v(x,y) = 0$ is considered analogously.

3. Solution. Because

$$\sin z = z + z^3/3! + \ldots = z(1 + z^2/3! + \ldots),$$

for $z \neq 0$ sufficiently small one has

$$\frac{1}{\sin z} = \frac{1}{z}(1 - z^2/3! + \ldots),$$

which is easily seen by the method of dividing power series. Therefore

$$f(z) = -z/3! + \ldots$$

admits a Taylor expansion in a neighborhood of $z = 0$ with constant term 0. So it defines a function which is analytic in a neghborhood of 0 if we put $f(0) = 0$.

4. Answer: $1 - z + 3z^2/2$.

5. Proof. Consider the Taylor expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

$$f(z^2) = \sum_{n=0}^{\infty} a_k z^{2k}.$$

Looking at $f(z) = z + f(z^2)$ one gets that $a_1 = 1$, $a_k = a_{2k}$ if k is even, and $a_k = 0$ if $k > 1$ is odd. Every integer k can be written in the form $k = 2^n p$ where p is not divisible by 2. Using the above equations one obtains

$$a_{2^n p} = a_{2^{n-1}p} = \ldots = a_{2p} = a_p = \quad 0 \text{ if } p > 1, \quad 1 \text{ if } p = 1.$$

6. Answer: $-1/16$.

7. Answer: $-\pi i/2$.

8. Answer: $\frac{\pi}{2b^3}(1+ab)e^{-ab}$.

9. Answer: 1.

10. Proof. By Cauchy formula,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz.$$

Parametrize C by $z(\phi) = a + Re^{i\phi}$, then

$$z'(\phi) = iRe^{i\phi}$$

and

$$z(\phi) - a = Re^{i\phi}.$$

Hence

$$\int_C \frac{f(z)}{z-a}dz = \int_0^{2\pi} \frac{f(a+Re^{i\phi})}{Re^{i\phi}} iRe^{i\phi} d\phi = i\int_0^{2\pi} f(a+Re^{i\phi})d\phi.$$