Math 185 Complex Analysis, Fall 2017. Instructor: Dmitry Tonkonog

Final

Duration: 3 hours

No books, notes or calculators are allowed. Several blank pages are provided at the end of the exam. If you need more paper, please get it from the proctor. In the case of a fire alarm, leave your exams in the room, face down, before evacuating. Under no circumstances should you take the exam with you.

This final has 10 problems.

Your name:

1:___/4 2:___/6 3:___/8 4:___/6 5:___/8

6:___/4 7:___/6 8:___/6 9:___/4 10:___/8 Σ:___/60

Problem 1 (4 points). Compute the real part of the complex number

$$(-4+3i)^3.$$

Problem 2 (6 points). Suppose $f(z)$ is an analytic function on a domain D and for every $z \in D$ it holds that

$$\operatorname{Im} f(z) = (\operatorname{Re} f(z))^2.$$

Prove that $f(z)$ is constant on D.

Problem 3 (8 points). For two complex numbers $a, b \in \mathbb{C}$, consider the function defined by

$$f(z) = \begin{cases} \dfrac{(\sin z) - a}{z - \pi/4}, & z \neq \pi/4, \\ b, & z = \pi/4. \end{cases}$$

(a) Find the numbers a, b for which $f(z)$ is an entire function.

(b) Having found the numbers a, b turning $f(z)$ into an entire function, consider its Taylor expansion $\sum_{n=0}^{\infty} a_n (z - \pi/4)^n$ at the point $z_0 = \pi/4$. Find the coefficients a_2 and a_3.

Hint: it may be helpful to use the trigonometric identity $sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

Problem 4 (6 points). Let $\sum_{n=0}^{\infty} a_n z^n$ be the Taylor expansion of

$$f(z) = \frac{1}{1 + 2\sin z}$$

at the origin. Find a_0, a_1, a_2.

Problem 5 (8 points). Suppose that $f(z)$ is analytic in a disk $|z| < \epsilon$ for some $\epsilon > 0$ and satisfies

$$(1 + z^2)f'(z) = 1$$

for all z such that $|z| < \epsilon$. Find the Taylor expansion of $f(z)$ at the origin.

Problem 6 (4 points). For a given integer $n > 0$ and a complex number $a \in \mathbb{C}$, compute the residue

$$\operatorname*{Res}_{z=0} \frac{e^{az}}{z^n}.$$

Problem 7 (6 points). Compute the integral

$$\int_C \frac{\sin(\pi z/2)}{(z^3 - z)(z - i)} dz$$

where the positively oriented contour C is:

(a) the circle $|z - 1| < 1$;

(b) the circle $|z - 1| < 4/3$.

Problem 8 (6 points). Let $a > 0$ be a real number. Compute the integral

$$\int_{-\infty}^{+\infty} \frac{x \sin x}{(x^2 + a^2)^2}$$

Problem 9 (4 points). Consider the polynomial

$$z^3 - 2z + 2.$$

Find the number of its roots within the disk $|z| < 2$, counted with multiplicities.

Problem 10 (8 points). Suppose a function $f(z)$ is analytic in the region $|z| \leq 1$,

$$f(0) = 0,$$

and

$$|f(z)| \leq 1 \quad \text{for all } z \quad \text{such that} \quad |z| \leq 1.$$

(a) Prove that

$$|f(z)| \leq |z| \quad \text{for all } z \quad \text{such that} \quad |z| \leq 1.$$

(b) Prove that there exists no function $f(z)$ which has all the properties stated above, and additionally satisfies

$$f'(0) = 2.$$

Hints: consider the function $g(z) = f(z)/z$ and argue that $g(z)$ can be assigned a value at the point $z = 0$ in a way which makes $g(z)$ analytic in the region $|z| \leq 1$. Then apply the maximum modulus principle to $g(z)$.

[Blank page]

[Blank page]

[Blank page]