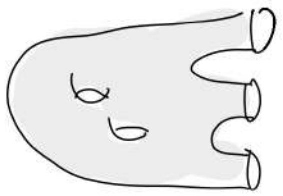
# lec 23 | An excursion: mapping class groups |

Great book: Farb, Margalit, 'A primer on mapping class groups'.

Let $\Sigma = \Sigma_{g,r}$ be a genus $g$ surface with $r$ boundary components.
(ie $\Sigma_{g,r} = \Sigma_g \setminus (r \text{ open disks})$)



$\Sigma_{2,3}$

<u>Def</u> $\mathrm{Diff}^+(\Sigma)$ = group of diffeos $\Sigma \to \Sigma$ preserving orientation

<u>Def</u> $\mathrm{Mod}(\Sigma) = \mathrm{Diff}^+(\Sigma) / \text{homotopy}$

This is a discrete group called the <u>mapping class group</u> of $\Sigma$.

<u>Note</u> Also have $\mathrm{Mod}(\Sigma) = \mathrm{Homeo}^+(\Sigma) / \text{homotopy}$

orient-preserving homeomorphisms

## Why Bother?

① Related to the study of 3-mflds.

Def A genus $g$ handlebody is the 3-mfd with bdy $H_g \subset \mathbb{R}^3$ with boundary the (standardly embedded) genus $g$ surface.

Solid inside

$$\partial H_g = \Sigma_g$$

## Construction (Heegaard splitting)

Let $f : \Sigma_g \to \Sigma_g$ be a diffeo, w
Take 2 disjoint copies of $H_g$ :

$$H_g^1 \quad \text{and} \quad H_g^2$$

Write $\partial H_g^1 = \Sigma_g^1$, $\partial H_g^2 = \Sigma_g^2$ and $f : \Sigma_g^1 \to \Sigma_g^2$ (think of $\Sigma_g^{1,2}$ as 2 copies of $\Sigma_g$).

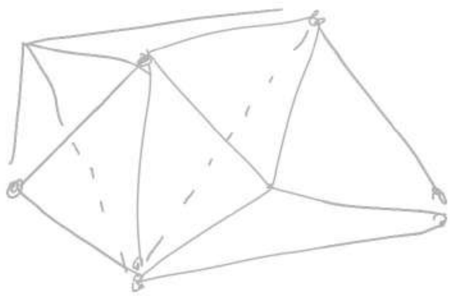
Then glue $H_g^1$ to $H_g^2$ along their boundaries via the map $f$, ie consider

$$M = H_g^1 \underset{f}{\cup} H_g^2 = \frac{H_g^1 \sqcup H_g^2}{(x \in \Sigma_g^1) \sim (f(x) \in \Sigma_g^2)}$$

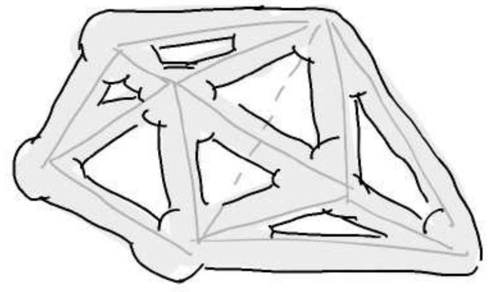
Then $M$ is a closed 3-mfd; we say $M$ has a Heegaard splitting given by $f$.

<u>Thm</u> (Heegaard) Every closed 3-mfd admits a Heegaard splitting, ie $\forall M^3$ $\exists g \in \mathbb{N}$, $f : \Sigma_g \to \Sigma_g$ s.t. $M = H_g^1 \underset{f}{\cup} H_g^2$.

<u>Proof sketch</u> $\forall$ 3-mfd admits a triangulation (subdivision into simplices, topologically).



Take $H' =$ Nbhood (1-skeleton)



Then $H' \underset{\text{diff}}{\cong} H_g'$, a handlebody

<u>Exercise</u> The complement $\overline{(M \setminus H_g^1)}$ is also diffeo$^c$ to a handlebody $H_g^2$;
and $M = H_g^1 \underset{\substack{\text{common}\\ \text{boundary}}}{\cup} H_g^2$.

$\square$

<u>Note</u> A non-trivial $f$ will be recovered once we identify $H_g^1, H_g^2$ both with the same std handlebody

The relation btw $\mathrm{Mod}(\Sigma_g)$ & 3-mfds is subtle (by no means close to bijection).

② Thurston's Virtually Fibered Conjecture (solved by Agol–Wise, 2012):

Any closed <u>hyperbolic</u> 3-mfd has a finite cover $N$ of the form:

$$N = \frac{\Sigma_g \times [0,1]}{\{(x,0) = (f(x),1)\}} \quad \text{for a diffeo } f: \Sigma_g \to \Sigma_g$$

this construction is called "mapping cylinder".
Here, "virtually" = "after a finite cover",
"fibered" = because $N$ has a fibration over $S^1$ with fibres genus $g$ surfaces.

③ Connections with 4-mfds (via Lefschetz fibrations)

④ Beautiful relations to hyperbolic geom.

<u>Examples</u> $\mathrm{Mod}(S^2) = \{1\}$

$\mathrm{Mod}(T^2) = SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \begin{array}{l} a,b,c,d \in \mathbb{Z} \\ ad-bc=1 \end{array} \right\}$

Indeed, a diffeo $T^2 \to T^2$

$\updownarrow$

diffeo $f(x,y): \mathbb{R}^2 \to \mathbb{R}^2$

s.t. $f(x,y) = f(x+1,y) = f(x,y+1)$

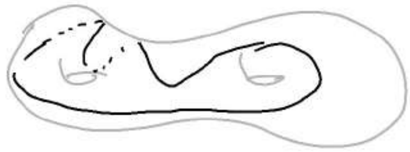
$\updownarrow$ htopy

linear map given by integral matrix.

(compare with the $S^1$ case from exercises)

$\mathrm{Mod}(\Sigma_g)$ is much more complicated for $g>1$.

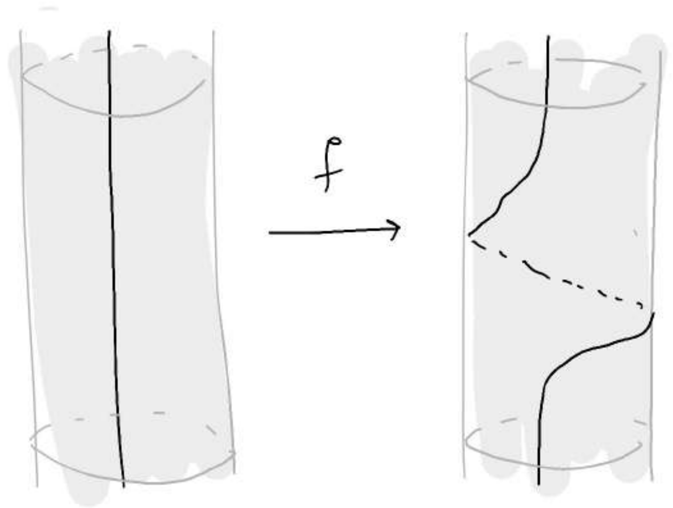
<u>Def</u> A simple closed curve (scc) in $\Sigma_g$ is an embedded $S^1 \subset \Sigma_g$



Def A model Dehn twist is a compactly-supported diffeo

$$S^1 \times \mathbb{R} \longrightarrow S^1 \times \mathbb{R}$$

given by the picture (modulo comp. supported htopy):

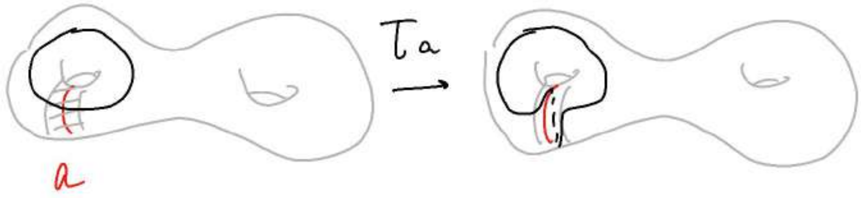


Def Let $a \subset \Sigma_g$ be a scc. The Dehn twist

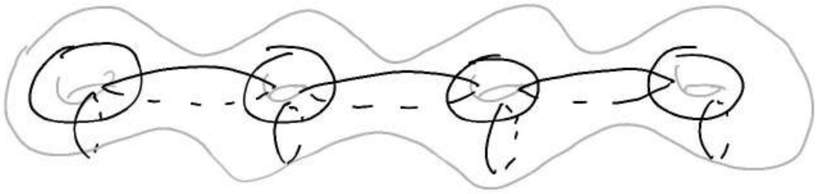
$$T_a \in \mathrm{Mod}(\Sigma_g)$$

is the (htopy class) of the diffeo defined by:

- The std Dehn twist in $\mathrm{Nbhood}(a) \cong S^1 \times \mathbb{R}$
- Identity in $\Sigma_g \setminus \mathrm{Nbhood}(a)$



Thm (Dehn-Lickorish) $\mathrm{Mod}(\Sigma_g)$ is generated by the Dehn twists along these curves:



A full set of relations is known, but complicated
Examples of relations:

- $T_a T_b = T_b T_a$ if $a \cap b = \emptyset$
- $T_a T_b T_a = T_b T_a T_b$ if $|a \cap b| = 1$

...

Thm (Dehn-Nielsen) $\mathrm{Mod}(\Sigma_g) \underset{\text{isom}}{\cong} \mathrm{Out}(\pi_1 \Sigma_g) =$

$= \mathrm{Aut}(\Sigma_g) / \{\text{conjugations}\}$

proof uses hyperbolic geometry!

Fundamental question: $f \in \mathrm{Mod}(\Sigma_g)$, what is its "nicest representative" by a diffeo $\varphi: \Sigma_g \to \Sigma_g$ in htpy class $f$?

Thm (Nielsen-Thurston classification).

Let $f \in \mathrm{Mod}(\Sigma_{g,r})$, then $\exists$ diffeo $\varphi : \Sigma_{g,r} \to \Sigma_{g,r}$ in htopy class $f$ such that one of the following holds:

- $\varphi$ is <u>periodic</u> : $\varphi^k = \mathrm{Id}$
- $\varphi$ is <u>pseudo-Anosov</u> (def'n is a bit involved)
- $\varphi$ is <u>reducible</u> <u>ie</u> fixes some (nonempty) collection of non-contractible closed curves $c_i$ s.t. $\varphi(c_i) = c_i$ & $\varphi|_{c_i} = \mathrm{Id}$

In the reducible case, cut along $c_i$ (this decreases $g$) & apply the thm again.

<u>What it says for $T^2$</u> Suppose $A \in SL(2;\mathbb{Z}) = \mathrm{Mod}(T^2)$

Also recall: $SL(2;\mathbb{Z}) < \mathrm{Isom}(\mathbb{H}^2)$

<u>Periodic</u> $\longleftrightarrow$ $\mathrm{tr}\, A = 0$ or $1$ <u>eg</u> $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$

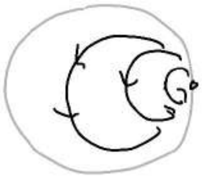
$\longleftrightarrow$ $A$ has finite order.

$\longleftrightarrow$ <u>elliptic</u> isometry of $\mathbb{H}^2$

(fixes one point in $\mathbb{H}^2$)

Reducible $\longleftrightarrow$ $\operatorname{tr} A = 2$ eg $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ (Dehn twist)

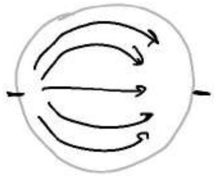
$\longleftrightarrow$ Parabolic isometry of $\mathbb{H}^2$



(fixes one pt in $\partial \mathbb{H}^2$)

(Pseudo-) Anosov $\longleftrightarrow$ $\operatorname{tr} A \geq 3$ eg $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$
Arnold's cat map

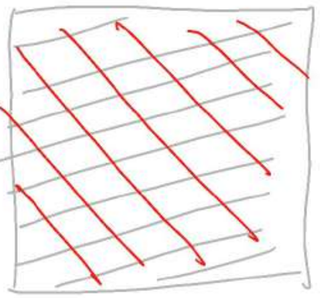
$\longleftrightarrow$ Hyperbolic isometry of $\mathbb{H}^2$



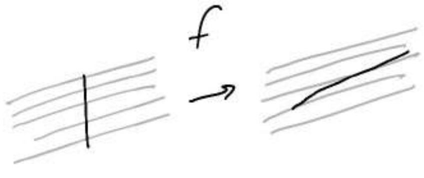
(fixes 2 pts in $\partial \mathbb{H}^2$)

In terms of eigenvalues $\lambda, \lambda^{-1}$ of $A$:
Periodic $\longleftrightarrow$ $\lambda, \lambda^{-1}$ are complex
Reducible $\longleftrightarrow$ $\lambda, \lambda^{-1} = \pm 1$
Anosov $\longleftrightarrow$ $\lambda, \lambda^{-1}$ are real

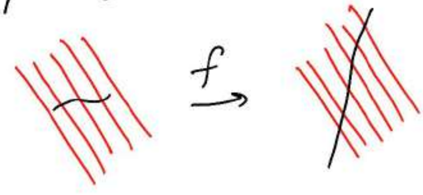
Let $A \in SL(2;\mathbb{Z})$ be Anosov, then the corresponding linear diffeo $f: T^2 \to T^2$ preserves 2 foliations on $T^2$ given by the lines in the eigenvector directions of $A$:



$f$ contracts in directions transverse to:



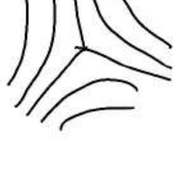

& expands in directions transverse to:




This combination creates <u>chaotic dynamics</u> of the iterations $f^k$. Compare/google: <u>Arnold's cat map</u>.

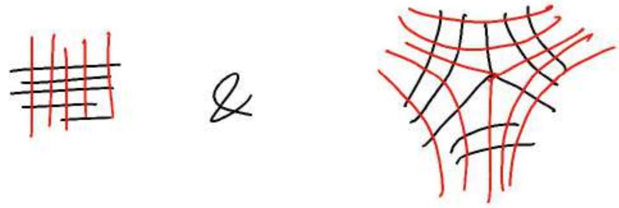
<u>Def</u> (due to Thurston, sketch)
A <u>pseudo-Anosov</u> diffeo $\varphi : \Sigma_g \to \Sigma_g$ is a diffeo which preserves two <u>transverse</u> <u>measured foliations</u> on $\Sigma_g$ & is shrinking resp. expanding the transverse measure of the first resp. second foliation by a factor of $\lambda$.

<u>Foliation</u> locally looks like  (up to diffeo), but is allowed to have singularities:



(must occur, for topology reasons)

Transverse foliations look like, locally & up to diffeo:



Pseudo-Anosov maps are chaotic; a "generic" mapping class is pseudo-Anosov.

Even for finite mapping classes, the theorem is non-trivial:

Thm (Nielsen) Suppose $f : \Sigma_g \to \Sigma_g$ is a diffeo & $f^K \underset{\text{htopic}}{\sim} Id$, then $\exists g : \Sigma_g \to \Sigma_g$,

$f \sim g$ such that $g^K = Id$.

(ie $f$ is a finite order map).

Teichmüller space (idea).

Teich $(\Sigma_g)$ is the space of hyperbolic metrics on $\Sigma_g$ up to the action of diffeo's htopic to $Id$,

or equivalently an element of Teich $(\Sigma_g)$ is $(X, \varphi)$ where

X is a genus g surface with a fixed hyperbolic metric,

$\varphi : \Sigma_g \to X$ is a diffeo,

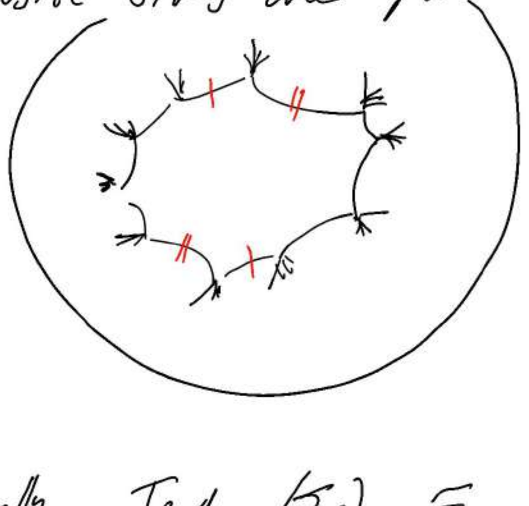
up to the relation $(\varphi, X) \sim (\varphi', X')$ if $\exists$ diagram

$$\begin{tikzcd} \Sigma_g \arrow[r, "\varphi"] \arrow[dr, "\varphi'"'] & X \arrow[d, "\text{isometry}"] \\ & X' \end{tikzcd}$$

commutative up to homotopy

<u>Equivalently</u> Teich$(\Sigma_g)$ is the space parametrising ways of <u>tiling</u> $\mathbb{H}^2$ by $4g$-gons with appropriate conditions guaranteeing they factor to a genus $g$ surface:

opposite sides are parallel & equal



<u>Topologically</u> Teich$(\Sigma_g) \underset{\text{homeo}}{\equiv} \mathbb{R}^{6g-6}$

but it has a natural metric which is almost hyperbolic (so think of $\mathbb{H}^{6g-6}$).

Important properties ⓐ $Mod(\Sigma_g)$ acts on $Teich(\Sigma_g)$

ⓑ Any isometry $X \to X$ isotopic to $Id$ must be equal to $Id$
hyperbolic

(follows from complex analysis: hyp. isometry $\Leftrightarrow$ cx map.)

Proof of Nielsen thm Suppose $k$ prime for simplicity

Let $[f] \in Mod(\Sigma_g)$ be the class of $f$, then $[f]^k = Id$ & $[f]$ induces a $\mathbb{Z}/k$ action on $Teich(\Sigma_g)$

Fact from topology There is no free $\mathbb{Z}/k$ action on a contractible finite-dim'l top space.

So $[f]$ fixes a point $(X, \varphi) \in Teich(\Sigma_g)$.

$$\begin{tikzcd} \Sigma_g \arrow[r, "\varphi"] \arrow[d, "f"'] & X \arrow[d, dashed, bend left] \\ \Sigma_g \arrow[r, "\varphi"'] & X \end{tikzcd}$$

$\exists$ isometry $g$ s.t. diagram commutes up to htopy

Claim $(\varphi^{-1} g \varphi)^k = Id$ (Note: $\varphi^{-1} g \varphi \sim f$).

Indeed: $(\varphi^{-1} g \varphi)^k \sim \mathrm{Id}$ because $[f]^k = \mathrm{Id}$,

<u>So</u> $g^k \sim \mathrm{Id}$ <u>But</u> $g$ is a hyperbolic isometry

<u>So</u> $g^k = \mathrm{Id}$

<u>So</u> $(\varphi^{-1} g \varphi)^k = \varphi^{-1} g^k \varphi = \mathrm{Id}$. $\square$