Lec 23 Pontryagin thy, finishing the proof.

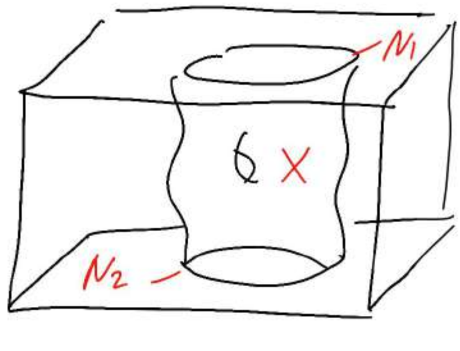
Let's recall what we did

Setup Study maps $f: M \to S^n$ up to htopy.

Pontryagin invariant $N = f^{-1}(p)$ reg. value with the framing obtained by $(df)^{-1}$(some fixed basis of $T_p S^n$), up to framed cobordism.

Recall: framed submfd $N \subset M$ is a submfd with a choice of a basis for $(T_x N)^\perp \subset T_x M$ for each $x \in N$, varying smoothly with $x \in N$.

Recall: a cobordism between $N_1, N_2 \subset M$ is a submfd $X \subset M \times [0,1]$ looking like this:



$M \times [0,1]$

A framed cobordism between framed submfds $N_1, N_2 \subset M$ is a cobordism such that the framings of N_1, N_2 extend to a framing of X.

Example $f_1, f_2 : M \to S^n$ htopic via htopy F, $p \in S^n$ is reg for f_1, f_2, F then

$f_1^{-1}(p)$ & $f_2^{-1}(p)$ are framed colordant via $F^{-1}(p)$ with the framing pulled back from a basis of $T_p S^n$.

Theorem 1 Two maps $f_1, f_2 : M \to S^n$ are htopic $\Leftrightarrow$ their Pontryagin invariants are the same (up to framed colordism)

Theorem 2 Any compact framed submfd $(N, \underset{\text{framing}}{\mathfrak{z}}) \subset M$ arises as the Pontryagin invt of a map $M \to S^n$, $n = \dim M - \dim N$.

Last time, we did the following three main steps.

① Proved Theorem 2. Recall the idea.
use product nbhood theorem:

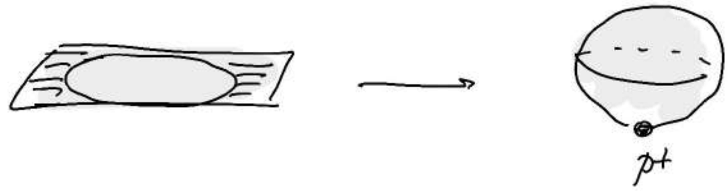
for any <u>framed</u> $(N, s) \subset M$, $\exists$ diffeo

$$\mathrm{Nbhd}(N \text{ in } M) \xrightarrow{\varphi} N \times \mathbb{R}^n$$

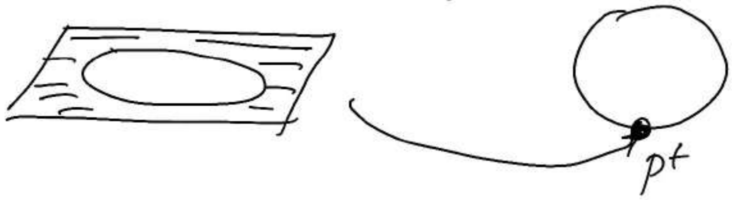
such that:

The given framing $s \xmapsto{d\varphi}$ const. basis in $\mathbb{R}^k$

<u>Then</u> use a standard map $\mathbb{R}^n \xrightarrow{\psi} S^n$



which sends everything away from a compact region to pt



& is a diffeomorphism to the complement:

Then use $\mathrm{Nbhd}(N \text{ in } M) \xrightarrow{\varphi} N \times \mathbb{R}^n \xrightarrow{\psi} S^n$

& extend by <u>const</u> to the rest of M.
"the nbhood of N hugs the sphere".

② Proved $\Rightarrow$ from Thm 1

(htopic $\Rightarrow$ same Pontryagin invts). This is quite straightforward

③ Proved an important lemma:

<u>Lemma</u> If $f_1, f_2 : M \to S^n$ have <u>identically the same</u> preimages $f_1^{-1}(p)$, $f_2^{-1}(p)$ <u>and</u> their framings <u>coincide</u>, then f_1 & f_2 are htopic.

Idea: because $f_1^{-1}(p) = f_2^{-1}(p) = N$ & framings coincide, can arrange that

$$f_1 \equiv f_2 \text{ in nbhood of } N$$

(using Product nbhd thm)

<u>Then</u> can homotop f_1 to f_2 away from N <u>using</u> the fact that $S^n \setminus \{p\}$ is contractible; we do this keeping f_1, f_2 fixed in nbhd of N.

Now we finish the proof of Thm!

<u>Proving</u> that f_1, f_2 have framed cobordant Pontr. mfds
$\Rightarrow$ f_1, f_2 are htopic

Let $(X, \mathcal{S}) \subset M \times [0,1]$ be a framed cobordism between $f_1^{-1}(p)$ & $f_2^{-1}(p)$ where $\mathcal{S}$ is a framing on X.

By Thm 2 applied to $(X, S) \subset M \times [0,1]$,

$\exists$ a map $F : X \times [0,1] \longrightarrow S^n$
whose Pontryagin framed mfd $F^{-1}(p)$ coincides with X.

Denote $\tilde{f}_0 = F|_{X \times \{0\}}$

$\tilde{f}_1 = F|_{X \times \{1\}}$

then:

$f_0 \underset{\text{htopic}}{\sim} \tilde{f}_0$, $f_1 \sim \tilde{f}_1$

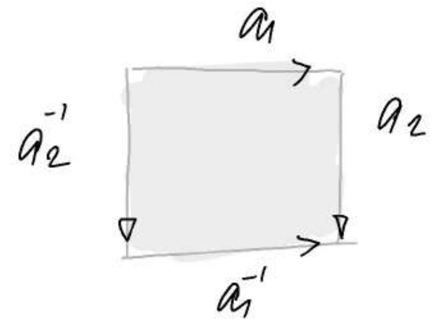
by the main Lemma above, because their Pontryagin mfds <u>coincide identically</u>.

Finally, $\tilde{f}_0 \underset{\text{htopic}}{\sim} \tilde{f}_1$ via F. $\square$

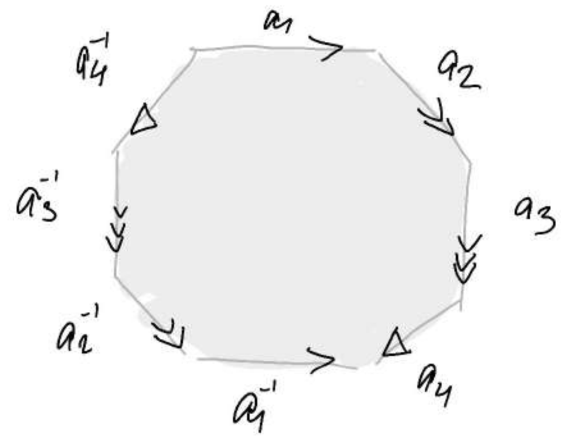
Fundamental groups of surfaces & Dehn's algorithm

Gluing the genus g surface

One can obtain the genus g surface Σ_g by gluing the sides of a $4g$-gon pairwise according to the following picture:



$g=1$



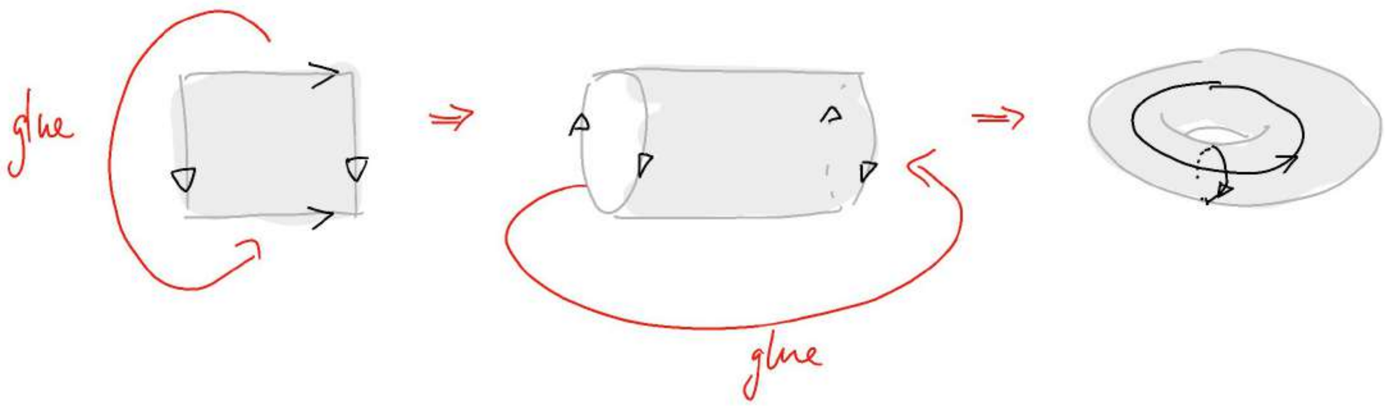
$g=2$

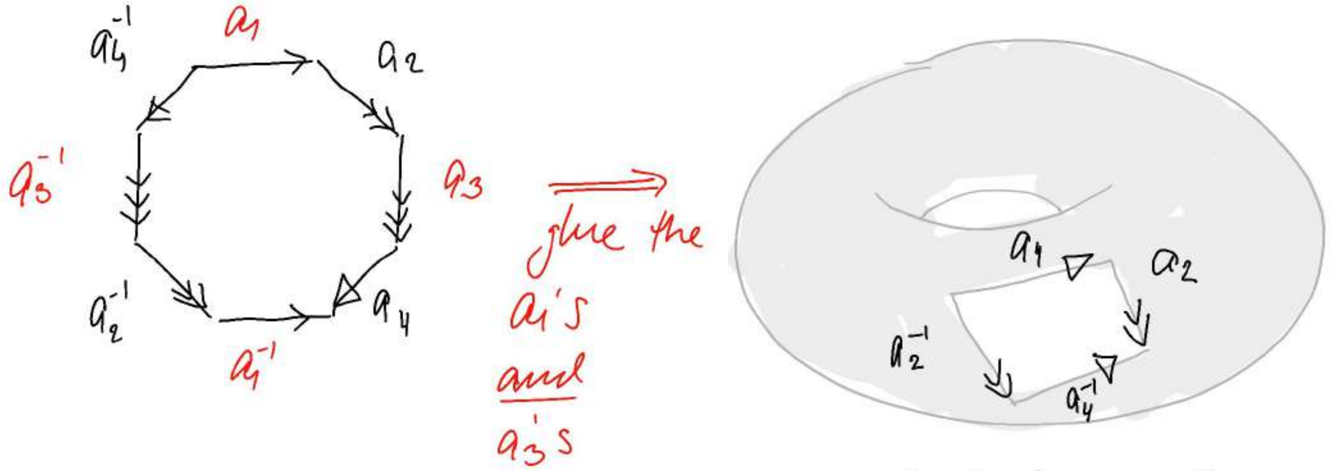
In general: $4g$-gon with arrows labelled:

$$\underbrace{a_1, \dots, a_{2g}}_{\text{oriented clockwise}}, \underbrace{a_1^{-1}, \dots, a_{2g}^{-1}}_{\text{oriented counterclockwise}}$$

the power -1 is for orientation counter clockwise.

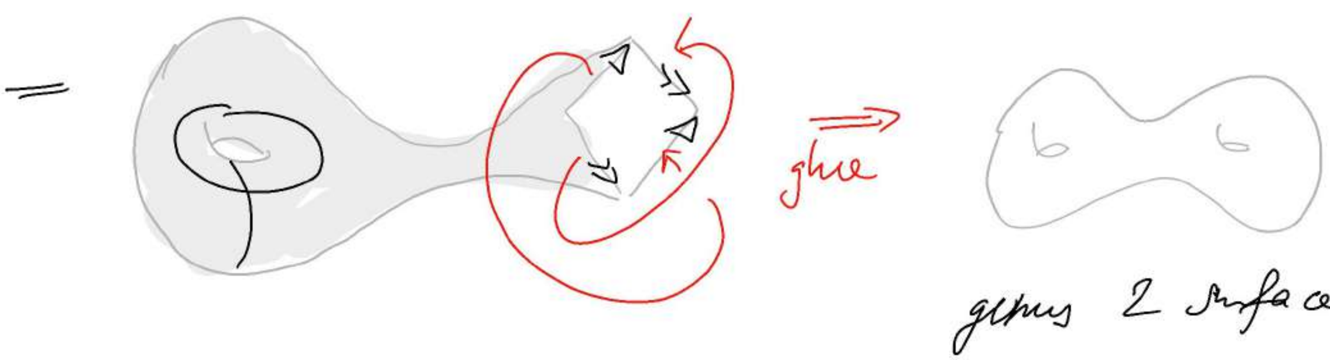
How to glue On a picture:





=

get 2-torus minus a square

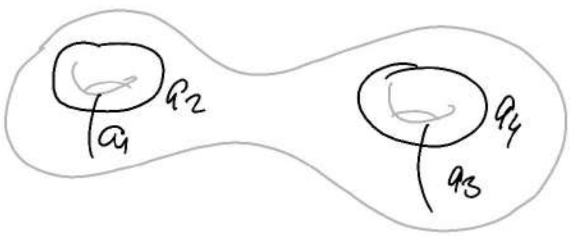


genus 2 surface

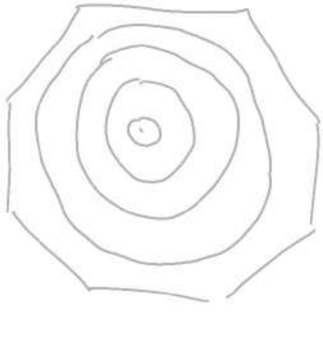
$$\underline{\text{Corollary}} \quad \pi_1 \Sigma_g \cong \langle \overset{\text{generators}}{a_1, \dots, a_{2g}} \mid \overset{\text{single relation}}{a_1 \dots a_{2g}\, a_1^{-1} \dots a_{2g}^{-1} = 1} \rangle$$

$$\cong \langle a_1 \dots a_g \mid \underbrace{\underbrace{[a_1, a_2]}_{\text{commutator: } a_1 a_2 a_1^{-1} a_2^{-1}} \dots [a_{2g-1}, a_{2g}]}_{\text{product of commutators}} = 1 \rangle$$

<u>Proof</u> (sketch) Each side of the $4g$-gon a_i gives rise to a <u>loop</u> in Σ_g



& the whole boundary of the 4-gon is the loop $a_1 \dots a_{2g} a_1^{-1} \dots a_{2g}^{-1}$ which is <u>contractible</u> inside the 4-gon:



Eg $\pi_1 T^2 \cong \mathbb{Z}^2$ because it's generated by 2 elements a_1, a_2 and the relation says

$$a_1 a_2 a_1^{-1} a_2^{-1} = 1 \iff a_1 a_2 = a_2 a_1$$

(ie a_1, a_2 commute)

Generally One can define groups by generators and relations

$$\langle a_1, \ldots, a_k \mid R_1 \ldots R_e \rangle = G$$

where:

- $a_1, \ldots, a_k$ are abstract symbols ("letters of the alphabet)
- $R_1 \ldots R_e$ are words in the alphabet $a_1^{\pm 1}, \ldots, a_k^{\pm 1}$.

A word in $a_1^{\pm 1}, \ldots, a_k^{\pm 1}$ is any ordered sequence of symbols:

$$a_{i_1}^{\pm 1} \ldots a_{i_s}^{\pm 1} \qquad i_j \in \{1 \ldots k\}$$

where s is called the length of the word.

- Elements of G = words in $a_1^{\pm 1}, \dots, a_n^{\pm 1}$ up to inserting/deleting subwords given by relations:

$$\underbrace{\text{word1 word2}} \sim \underbrace{\text{word1}} \, R_1 \, \underbrace{\text{word2}}$$

<u>and</u> inserting/deleting $a_i a_i^{-1}$

- The $\emptyset$ word is the unit; inverse is

$$(a_{i_1} \dots a_{i_s})^{-1} = a_{i_s}^{-1} \dots a_{i_1}^{-1}$$

<u>Dehn's problem</u> Fix a group G given by generators & relations.
Does there exist an algorithm which receives, as input, a word in the generators, and answers whether the word represents the identity in G.

a word $w = a_{i_1}^{\pm 1} \dots a_{i_s}^{\pm 1}$

$\downarrow$

$\boxed{\text{Algorithm}}$

$\downarrow$

Yes/No: is $w = \emptyset$ in G?

<u>Thm</u> (Markov) There exists a group G for which Dehn's problem is algorithmically unsolvable

(reduces to the halting problem for Turing machines). There are plenty of such groups!

<u>Idea</u> why such examples exist. Suppose $w \sim \emptyset$ in G; this means that w can be brought to the empty word by deleting <u>or</u> <u>inserting</u> relations.

Sometimes, before deleting the relations, we may have to insert some & increase the length of w!

There may be no apriori estimate on how long w needs to become before it takes the form w':

$$\underset{\text{add relations}}{w \longrightarrow} w' = \underbrace{R_{i_1}^{\pm 1} \dots R_{i_p}^{\pm 1}}_{\text{relations}} \underset{\text{cancel relations}}{\longrightarrow} \emptyset$$

<u>Theorem</u> (Dehn) Dehn's problem is solvable for $\pi_1 \Sigma_g$. In fact, when $g \geq 2$, if a word

$$w \in \Sigma_g$$

represents the identity, then it can be brought to $\emptyset$ by <u>never increasing its length</u>.

<u>Dehn's lemma</u> Suppose w is a word in $a_1^{\pm 1}, \dots, a_{2g}^{\pm 1}$ and $w \sim \emptyset$ in $\pi_1 \Sigma_g$.

Then w contains a subword of length $\geq 2g+1$ which is also a subword of the only relation

$$R = \underbrace{a_1 a_2 \dots a_{2g} a_1^{-1} \dots}_{\text{a subword}} a_{2g}^{-1}$$

Then we can <u>shorten</u> w using R & reiterate using Dehn's lemma, thus solving Dehn's problem.

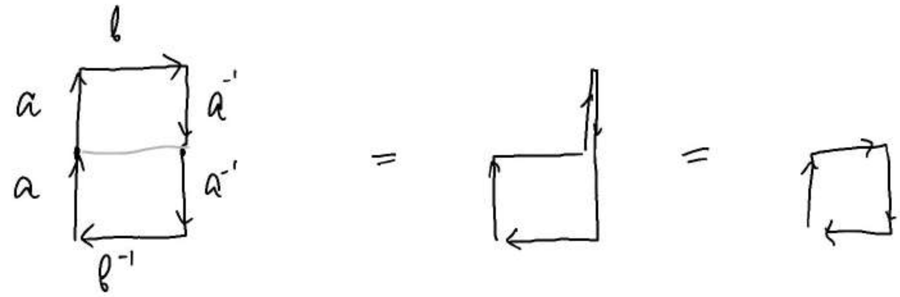
We will prove Dehn's lemma shortly. First, compare with the Dehn problem for $\pi_1 T^2 = \mathbb{Z}^2 =$
$= \langle a, b \mid ab = ba \rangle$

Here, we know the solution to Dehn's problem.

$$w = a^{p_1} b^{q_1} a^{p_2} b^{q_2} \dots a^{p_n} b^{q_n}$$

$w \sim \emptyset$ iff $\Sigma p_i = 0$ & $\Sigma q_i = 0$.

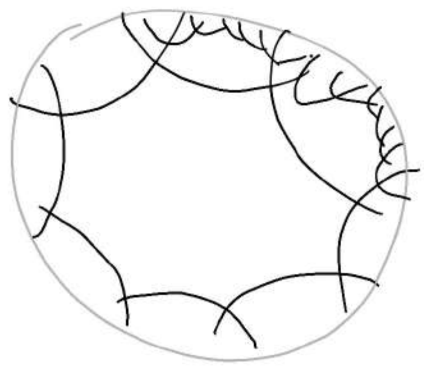
Here, Dehn's lemma is in fact wrong: sometimes need to <u>keep</u> the length of w (not decrease it) if we want to cancel w using $aba^{-1}b^{-1} = 1$.

Eg 

$w = aaba^{-1}a^{-1}b^{-1} = aba a^{-1}a^{-1}b - aba^{-1}b^{-1} = 1$

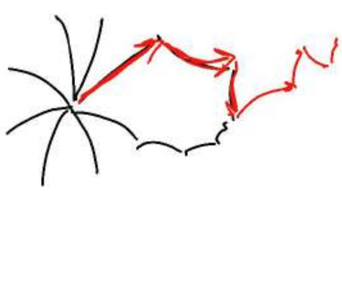
At this step, we kept the length.

<u>Proof of Dehn's lemma</u> When $g \geq 2$, can tile the hyperbolic disk by regular $4g$-gons.



(google it)

A word in $a_1^{\pm 1}, \ldots, a_{2g}^{\pm 1}$ gives a <u>path</u> along the edges of the tiling, once a starting point (vertex) is fixed:



$w \not\sim \emptyset \iff$ it gives a triv. elet in $\pi_1 \Sigma_g$

$\iff$ the path in $\mathbb{H}^2$ we get is actually <u>closed</u> (the endpoint coincides with the starting point)

<u>Dehn's lemma, reformulated</u> Consider a closed path along the edges of the $4g$-gonal tiling of $\mathbb{H}^2$.

Then it contains $\geq 2g+1$ consecutive edges of some $4g$-gon.

Proof: exercise