lec 20

# Hopf degree theorem, and the Pontryagin construction

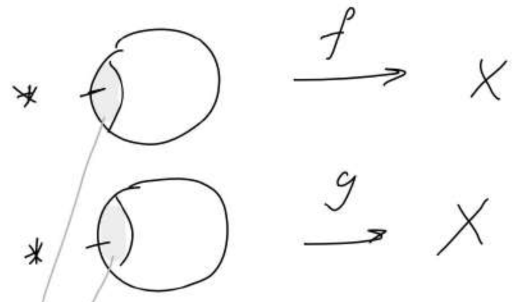
## Context: homotopy groups

Def let $X$ be a topo. space, $* \in X$ a "basepoint". The n'th homotopy group of $X$ is:

$$\pi_n(X) = \{ \underset{* \longmapsto *}{S^n \longrightarrow X} \} / \text{htopy}$$

chosen basepoints on $S^n$ & $X$ must be preserved

The group structure is defined as follows:




Apply htpy to make $f, g$ const in the nbhood;

then do connected sum:


Notation: $f+g$ or $f \# g$

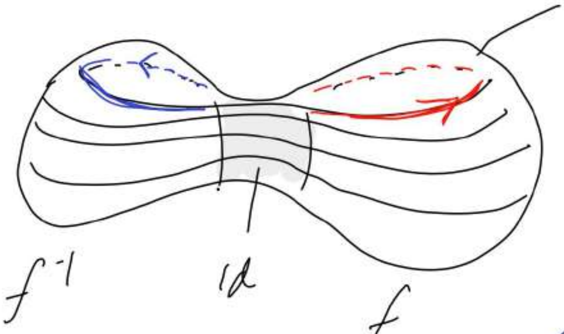
Up to htpy, this is well-defined

Inverse is given by the composition:

$(-f)$ : $S^1 \xrightarrow[\text{orient rev.}]{} S^1 \xrightarrow{f} X$

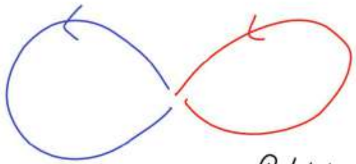
any orient-reversing map eg reflection

<u>Indeed</u>



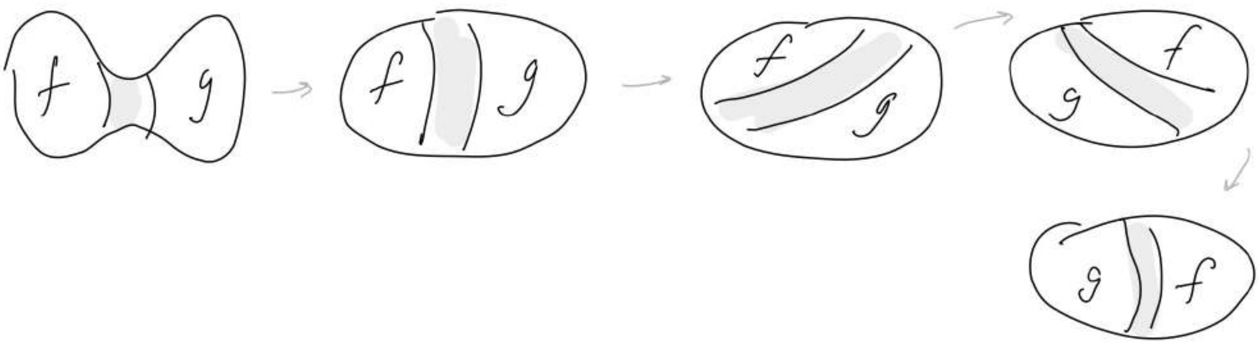

for each circle like this foliating $S^n$,

I get:




So can cancel $f, f^{-1}$ on each loop by reparametrising.

For $n \geq 2$, <u>$\pi_n(X)$ is commutative</u>:



- $\pi_1(S^1) = \mathbb{Z}$, $\pi_n(S^1) = 0$ for $n \geq 2$

- $\pi_k(S^n) = 0$ for $k < n$

- $\pi_n(S^n) = \mathbb{Z}$ — follows from Hopf theorem!

- $\pi_{n+1}(S^n) = \mathbb{Z}/2$ except: $\pi_3(S^2) = \mathbb{Z}$, $\pi_3(S^1) = 0$

- $\pi_{n+2}(S^n) = \mathbb{Z}/2$ except $\pi_3(S^1) = 0$

- Higher homotopy groups of spheres: homotopy chaos!

<u>Hopf theorem</u> If $M^n$ is closed, orientable then a map $M^n \longrightarrow S^n$ is determined by its degree, up to htopy.

Corollary $\pi_n S^n \xrightarrow[\cong]{\deg} \mathbb{Z}$.

Will follow the cobordism approach from Milnor.

<u>Def</u> Let $N, N' \subset M$ without bdy. They are <u>cobordant in $M$</u> if

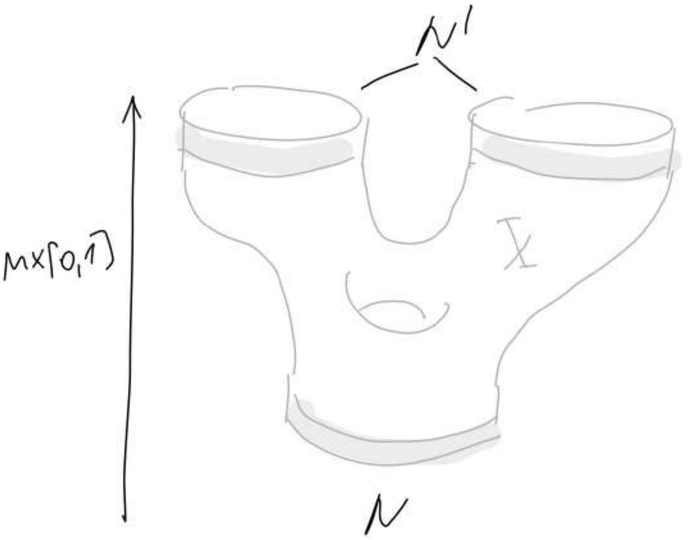
$\exists$ comp mfd $\quad X \subset M \times [0,1]$ $\quad$ (<u>cobordism</u>)

such that

$$\partial X = N \times \{0\} \cup N' \times \{1\},$$

$X$ does not intersect $M \times 0$, $M \times 1$ away from $\partial X$, and locally near $\partial X$, $X$ looks like:

$$N \times [0,\varepsilon) \quad \text{and} \quad N' \times (1-\varepsilon, 1].$$



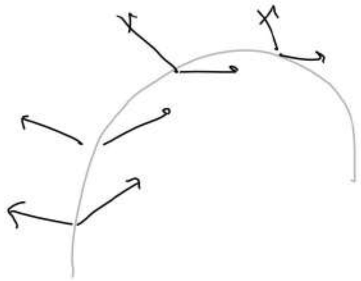

$N' = 2$ circles

$N =$ circle

<u>Def</u> A <u>framing</u> s of $N^n \subset M^m$ is a choice of $m-n$ sections $s_1 \dots s_{m-n}$ of $N_M N$ - normal bdle of $N$

such that $\forall x \in N$, $s_1(x), \dots, s_{m-n}(x)$ form a basis of the v.space $(N_M N)_x$.

In other words: $s$ is a smooth choice of a basis for $(N_M N)_x$ for each $x$.



Framing of a curve in $\mathbb{R}^3$

The pair $(N, s)$ is called a framed submfd

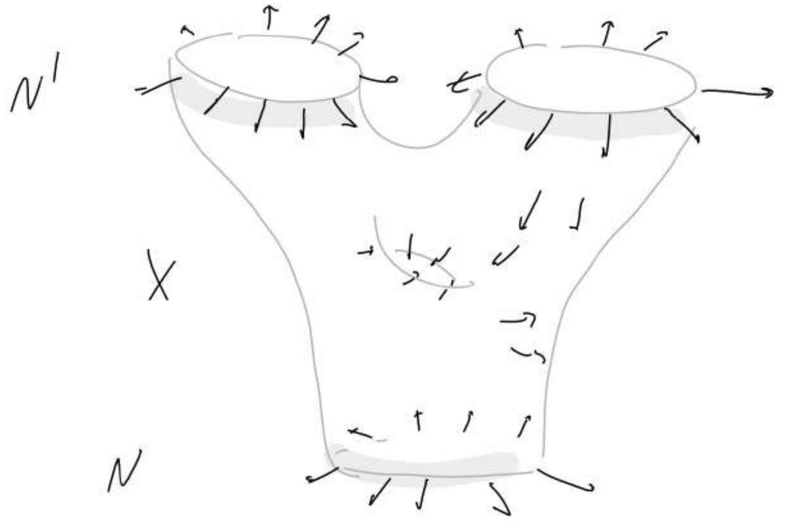
Note Not all $N \subset M$ have a framing — for it to exist, $N_M N$ must be trivial, ie diffeoc to $N \times \mathbb{R}^{m-n}$ !

Def Two framed submfds are framed cobordant if $\exists$ cobordism $X \subset M \times [0,1]$ btw $N, N'$ and a framing $u$ of $X$ such that:

$$u_i(x,t) = (s_i(x), 0) \quad \text{on } (x,t) \in N \times [0, \varepsilon)$$

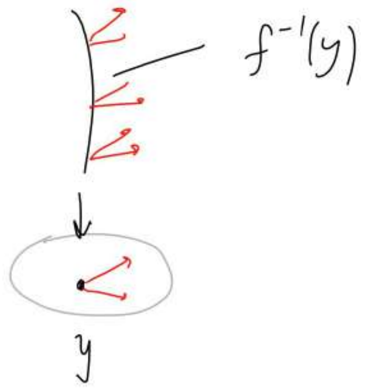
$$u_i(x,t) = (s_i'(x), 0) \quad \text{on } (x,t) \in N \times (\varepsilon-1, 1]$$

framing for $N$ resp. $N'$




— framing is "horizontal" and coincides with one on $N'$ here

<u>Now</u> let $f: M \to S^p$ be a smooth map, $y \in S^p$ reg. value, denote $N = f^{-1}(y)$




Recall: $\forall x \in f^{-1}(y)$, $\quad N_M N \xrightarrow[\cong]{df_x} T_y S^p$
isomorphism

<u>So</u> if we fix a pos. oriented basis in $T_y S^p$: $u = (u_1 \dots u_p)$ then

$$f^* u = \left( (df_x)^{-1} u_1, \dots, (df_x)^{-1} u_p \right) \in (N_M N)_x$$

is a framing for $N = f^{-1}(y) \subset M$.

Note In particular, the normal bdle to $N$ is trivial whenever $N = f^{-1}(y)$ for some map $f$, and some point $y$ in the target.

Def The framed mfd $(f^{-1}(y), f^* u)$

is called the Pontryagin mfd associated with $f$.

Theorem A The framed cobordism class of the Pontryagin mfd is a htopy invariant of a map $X \longrightarrow S^p$

Theorem B The Pontryagin invariant is full: two maps $X \to S^p$ are htopic iff their Pontryagin mfds are framed cobordant.

Theorem C Any closed framed submfd $(N, s) \subset M$ of codim $p$ occurs as Pontryagin mfd for a smooth map $M \to S^p$.

Lemma 1 Two pos. oriented bases $u, u'$ for $T_y S^p$ give cobordant Pontryagin mfds

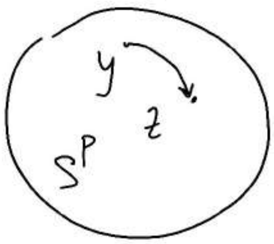
$$(f^{-1}(y), f^* u), \qquad (f^{-1}(y), f^* u')$$

**Proof** Choose a path of pos. oriented bases btw. $u \rightsquigarrow u'$
& pullback by $df$ $\square$

**Lemma 2** If $y$ reg value of $f$, $z$ suff. close to $y$ then $f^{-1}(z)$ is framed cobordant to $f^{-1}(y)$

framing implied but omitted from notation

**Proof** "Regular" is open condition so $\exists$ $\varepsilon$-nbhood of $y$ consisting of regular pts.




Consider rotations $r_t : S^p \to S^p$ :

- $r_t = Id \qquad 0 \le t \le 1/3$
- $r_t = r_1 \qquad 2/3 \le t \le 1$
- $r_t(y)$ moves on a great circle from $y$ to $z$

Define htpy $F : M \times [0,1] \to S^p$

$$F(x,t) = r_t f(x).$$

Note: $z$ is a reg value for $r_t \circ f : M \to S^p$

so $z$ is a reg value for $F$.

so $F^{-1}(z) \subset M \times [0,1]$ is a submfld;

it is canonically framed using $(dF)^{-1}$ & provides the desired framed cobordism $\square$

Lemma 3 If $f \underset{\text{isotopic}}{\sim} g$ & $y$ is reg value

for $f, g$ then $f^{-1}(y)$ is framed cob. to $g^{-1}(y)$

Proof Choose $F$ st

$F(x,t) \equiv f \qquad 0 < t < 1/3$

$F(x,t) \equiv g \qquad 2/3 < t < 1$

and $z \in \mathrm{Nbhd}(y)$ regular for $f, g$ and $F$.

Then $F^{-1}(z) \subset X \times [0,1]$ is the desired cobordism $\square$

Lemma 4 For $f : X \to S^p$, the framed cobordism class of $f^{-1}(y)$ does not depend on the choice of regular value $y$

Proof let $y, z$ be two reg values & $r : S^p \to S^p$ rotation,

then $f \sim r \circ f$ and

$$(r\circ f)^{-1}(z) = f^{-1}r^{-1}(z) = f^{-1}(y)$$

Theorem A follows. Move on to Thm B.

<u>Lemma 5</u> If $f, g : X \to S^p$ & $y \in S^p$ reg for $f$ & $g$ and $(f^{-1}(y), f^*u)$, $(g^{-1}(y), g^*u)$ are <u>equal</u> then:

$$f \sim g.$$

<u>Proof</u> Let $N = f^{-1}(y)$. The hypothesis means that

$$\forall x \in N, \quad df_x = dg_x.$$

<u>Step 1</u> Assume $f \equiv g$ in nbhood of $V$ of $N$.
Let

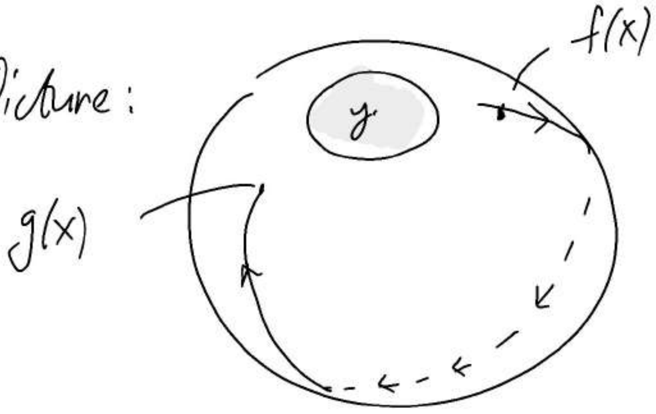
$$h : S^p \setminus \{y\} \to \mathbb{R}^p \qquad \text{Stereographic}$$

then:

$$F(x,t) = \begin{cases} f(x) & x \in V \\ h^{-1}[t\cdot h(f(x)) + (1-t)\cdot h(g(x))] & x \in M \setminus N \end{cases}$$

proves that $f \sim g$

Picture:



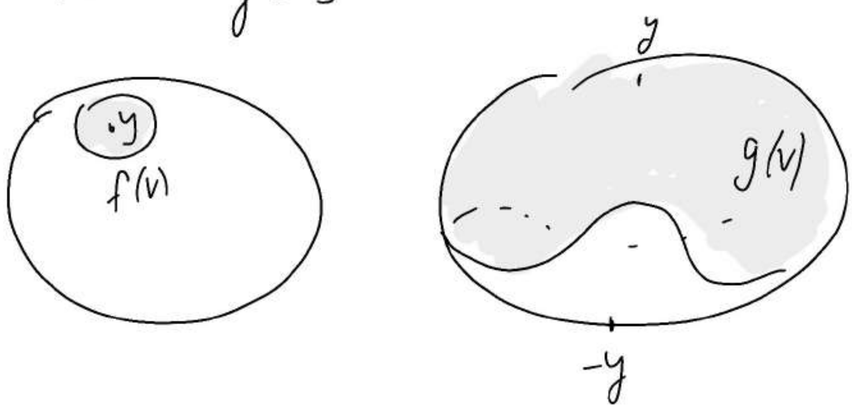

$f(x), g(x) \notin Nbhd(y)$
then can move
$f(x) \rightsquigarrow g(x)$
along great circle
missing $y$

Step 2 Now need to make $f \equiv g$ in nbhood.

Use product nbhood thm: given that $N_M N$ is trivial, we have: $\underset{\text{nbd of } N}{V} \cong N \times \mathbb{R}^P$.

[Note: formally we never proved this — see Milnor]

assume $V$ is small enough so that $g(V)$ does not contain $-y \in S^P$




Identify: $V = N \times \mathbb{R}^P$
$S^P \setminus \{-y\} = \mathbb{R}^P$

and get:

$$F, G: N \times \mathbb{R}^p \longrightarrow \mathbb{R}^p$$

coming from $f|_V$ and $g|_V$.

We have: $F^{-1}(0) = G^{-1}(0) = N \times 0$

and: $dF_{(x,0)} = dG_{(x,0)} = \text{std. proj.} \quad \forall x \in N.$

<u>Claim</u>

$\exists C: \ F(x,u)\cdot u > 0, \ G(x,u)\cdot u > 0$

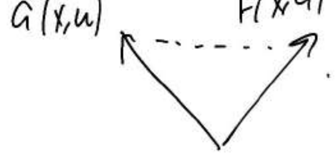
for $x \in N$, $0 < \|u\| < C$.

<u>ie</u> $F(x,u)$ & $G(x,u)$ are in the same half-space in $\mathbb{R}^p$.

Then the htopy

$$(1-t)F(x,u) + t\,G(x,u)$$

never hits $0$ away from $u = 0$




<u>Proof of claim</u> By Taylor:

$$\|F(x,u) - u\| = \bar{o}(\|u\|)$$

$$\| \quad dF_{(x,0)}(x,u)$$

<u>so</u> $|(F(x,u) - u)\cdot u| = \bar{o}(\|u\|^2)$

So $F(x,u)\cdot u = \|u\|^2 + \bar{o}(\|u\|^2) > 0$
for small $u$ □

∘ Modify our htpy above by cutoff

$$\lambda : \mathbb{R}^P \to \mathbb{R}, \qquad \lambda(u)=1, \ \|u\| \leq c/2$$
$$\lambda(u)=0, \ \|u\| \geq c$$

and consider homotopy:

$$F_t(x,u) = [1-\lambda(u)t]\,F(x,u) + \lambda(u)t\,G(x,u)$$

from $F = F_0$ to a map $F_1 : N \times \mathbb{R}^P \to \mathbb{R}^P$ s.that:

- ∘ $F_1 \equiv G$ when $\|u\| \leq c/2$
- ∘ $F_1 \equiv F$ when $\|u\| \geq c$
- ∘ has no zeroes.

Then use Step 1 to complete the lemma. □

Thm B follows.
Hopf degree theorem follows as well. In this case, $f^{-1}(y)$ is a bunch of points & framing $\longleftrightarrow$ sign $df_x$
The framed cobordism class is precisely computed by the degree!