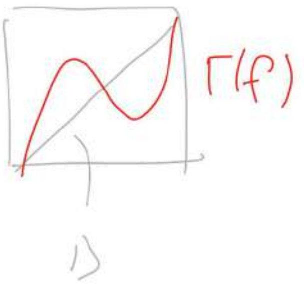
Lec 19 Last time:

Used intersec thy to define:

$\chi(X)$, Euler char of a mfd

$L(f)$, Lefschetz number of a map $f: X \to X$

Recall: $L(f) = I(\Delta, \Gamma(f))$



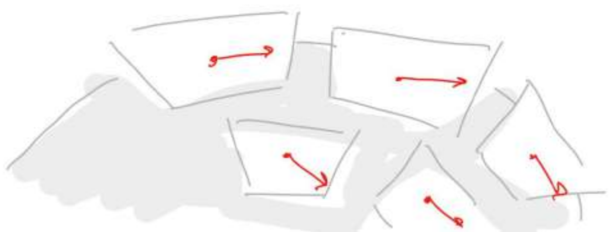
and $L(\mathrm{Id}) = \chi(X)$.

$\boxed{\text{Vector fields \& Poincare-Hopf thm}}$

Def A vector field on a mfd $X \subset \mathbb{R}^N$ is a smooth choice of one vector for each tangent plane $T_x X$, $x \in X$,

ie

a smooth map $\bar{v} : X \to \mathbb{R}^N$ s.t. $\bar{v}(x) \in T_x X$, $\forall x \in X$



Important to study: zeroes of $\bar{v}$, ie points $x \in X$ s.t. $\bar{v}(x) = 0$

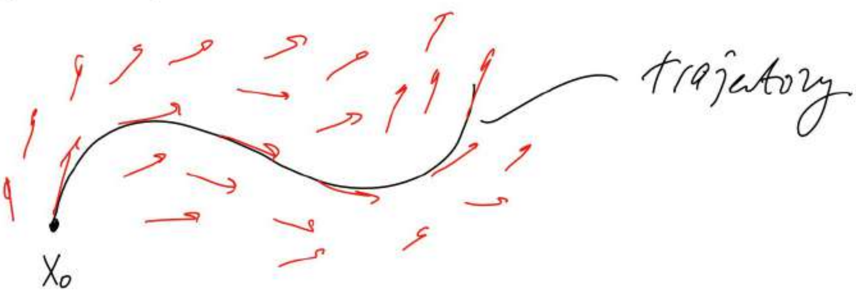
Dynamical interpretation Any vector field gives a 1parametric family of diffeomorphisms

$$\varphi_t : X \to X \qquad t \in \mathbb{R}$$

where $\varphi_0 = \mathrm{Id}$ and φ_t is the "time t flow of $\bar{v}$".
The trajectory of any point:

$$\varphi_t(x_0) \qquad t \in \mathbb{R}, \; x_0 \in X \text{ fixed}$$

is always tangent to $\bar{v}$:



Then zeroes of $\bar{v}$ $\Leftrightarrow$ points such that $\varphi_t(x) = x$, $\forall t$ (equilibria of our dynamical system φ_t).

Def let $x \in X$ be an isolated zero of v. Let B_ε be a small ball around x. The index of v at x is

$$\mathrm{ind}_x \bar{v} = \deg\left(x \mapsto \frac{v(x)}{|v(x)|} : \partial B_\varepsilon \to S^{n-1} \right)$$

where $n = \dim X$.

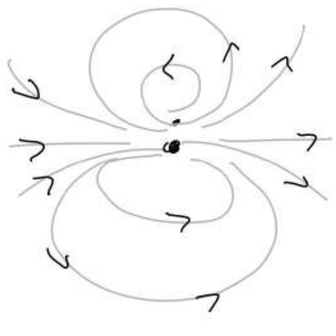
<u>Examples</u> Indices at the origin:

$+1$ $\quad$ $+1$ $\quad$ -1 $\quad$ -1

Why so? As I go around , I look at the tangent vector at the point I'm at.
For the left two pictures, the vectors rotate <u>counterclockwise</u> as I go ;

for the right two pictures, the vectors rotate <u>clockwise</u>.

<u>Ex</u>



Here index at 0 equals $+2$

Because: as I go , the vector field makes <u>two</u> full rotations counterclockwise

Technical remark

In the defn of index, when we write the map

$$\frac{\bar{v}(x)}{|\bar{v}(x)|} : \partial B_\varepsilon \to S^{n-1}$$

we assume that a parametrisation

$$\text{open} \ \begin{array}{c}\mathcal{U} \\ \cap \\ \mathbb{R}^n\end{array} \xrightarrow{\ \varphi\ } \text{Nbhd of } x \text{ in } X$$

and pullback v to the vector field

$$\varphi^* \bar{v} := (d\varphi)^{-1} \bar{v}$$

on $\mathcal{U} \subset \mathbb{R}^n$. Then we indeed have a map

$$\frac{\varphi^* v}{|\varphi^* v|} : \partial B_\varepsilon(0) \longrightarrow S^{n-1}.$$

We always assume that this is done but skip this detail for brevity. Note: without mentioning this,

$$\frac{v}{|v|} \text{ apriori lands in } S^{N-1} \text{ where } X \subset \mathbb{R}^N,$$

and we don't want this!

Poincaré-Hopf thm If $\bar{v}$ is a smooth vector field on X (comp., oriented) & $\bar{v}$ has isolated zeros, then

$$\chi(X) = \sum_{x : \bar{v}(x) = 0} \operatorname{ind}_x \bar{v}$$

Proof

Step 1 let f_t be the flow of $\bar{v}$.
We assume it for granted, but (for small t) one can define f_t in charts — so the existence of flows is a question about vector fields on $\mathbb{R}^n$ & is a subject of ODEs.
[There's a relatively explicit fla for f_t in terms of integrals]

Taking $f_t : X \to X$, the basic property of the flow is:

$$\forall x \in X, \quad \left.\frac{df_t(x)}{dt}\right|_{t=0} = \bar{v}(x) \in T_xX, \text{ or:}$$

$$\underbrace{f_t(x) = x}_{\text{because } f_0 = \mathrm{Id}} + t\,\bar{v}(x) + \bar{o}(t^2) \qquad (\star)$$

Claim If x_0 is an isolated zero of $\bar{v}$, then
$$\operatorname{ind}_{x_0} \bar{v} = \underbrace{L_{x_0}(f_t)}_{\text{local Lefschetz number defined last time}}, \text{ for any fixed } t > 0$$

(note: not assuming f_t is Lefschetz)

assuming f has no fixed points other than zeroes of $\bar{v}$.

Proof Use ($\ast$):

$$f(x) - x = t\bar{v}(x) + \bar{\bar{o}}(t^2)$$

$$\Downarrow$$

$$\frac{f(x) - x}{|f(x) - x|} = \frac{\bar{v}(x) + \bar{\bar{o}}(t)}{|\bar{v}(x) + \bar{\bar{o}}(t)|}$$

Now let x run through $\partial B_\varepsilon(x_0)$ where x_0 is an isolated zero of $\bar{v}$.

Then;

deg of this computes $L_{x_0}(f)$ (by defn)

same degree as for $\frac{\bar{v}(x)}{|\bar{v}(x)|}$ (by putting $t=0$)

& deg of this computes $\mathrm{ind}_{x_0} v$ $\square$

Step 2 Claim: for small enough t,

$\mathrm{Fix} f_t = \{\text{zeroes of } \bar{v}\}$

(we take it without proof).

Take such $f_t : X \to X$ then:

- $f_t \underset{\text{htopic}}{\simeq} \mathrm{Id}$ — clear
- $\chi(X) = L(f)$ — by Step 1 $\square$

Another point of view on $\chi(X) = \sum_{x : \bar{v}(x)=0} \operatorname{ind}_x \bar{v}$

Def Let $E \xrightarrow{\pi} X$ vector bundle, ie

$\forall$ nbhood $\mathcal{U} \subset X$, $\pi^{-1}(\mathcal{U}) \cong E \times \mathbb{R}^k$ & the diag commutes:

$$
\begin{tikzcd}
\pi^{-1}(\mathcal{U}) \arrow[rr, "\cong"] \arrow[dr, "\pi"'] & & \mathcal{U} \times \mathbb{R}^k \arrow[dl, "\text{std. projection}"] \\
& \mathcal{U} &
\end{tikzcd}
$$

For $p \in X$, we call $\pi^{-1}(p) = E_p \cong \mathbb{R}^k$ the fibre over p.

(Note: k is called the rank of vector Bdle E; not necessarily equal to $\dim X$!)

Def A section of E is a map s:

$$
\begin{tikzcd}
X \arrow[r, "s"] & E \arrow[l, bend left, "\pi"]
\end{tikzcd}
\quad \text{such that} \quad \pi \circ s = Id.
$$

This means $s(p) \in E_p \cong \mathbb{R}^k$ — so a section is a smooth assignment of a vector in the fibre E_p, for each p.

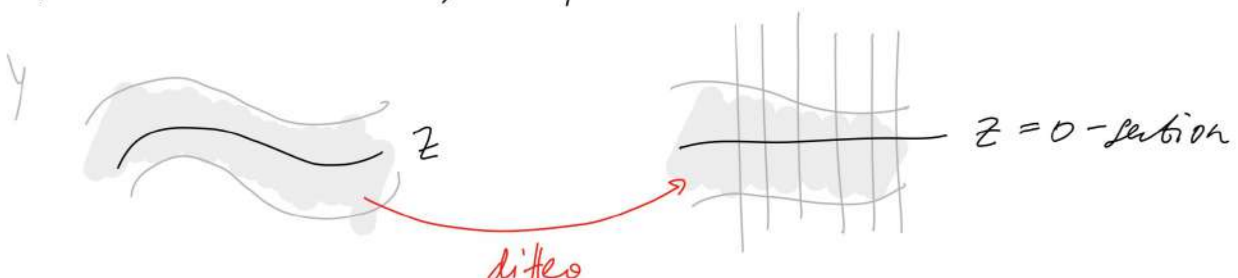
Example A vector field on X is the same thing as a section of TX.

Now Recall:

normal bdle

- $N_{X\times X}\,\Delta \cong TX$

- ε-nbhood thm: for $Z \subset Y$, an open nbhood of Z in Y is diffeoc to (some open nbhood of the 0-section in) $N_Y Z$



$Z \hookrightarrow Y$, as an embedding

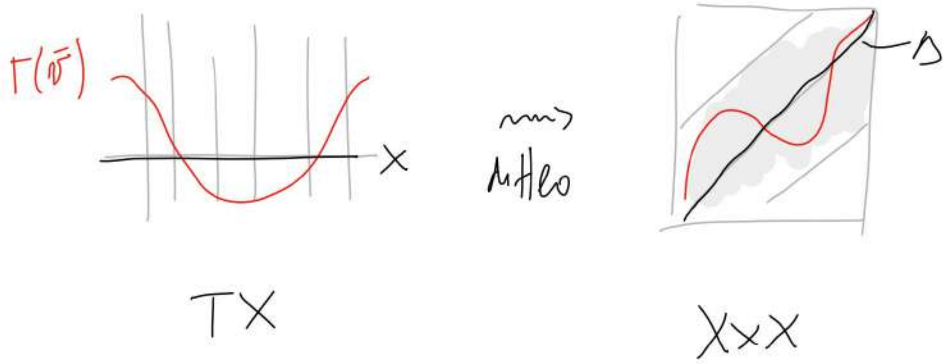
$N_Y Z$, as an abstract bdle over Z

- So: A nbhood of Δ in $X\times X$ is diffeoc to TX.

- A section $s: X \to E$ defines its <u>graph</u>
$\Gamma(s) \subset E$, $\quad \Gamma(s) = \{(x, s(x)) : x \in X, s(x) \in E_p\}$

- In particular the graph of a vector field on X is a submfd
$$\Gamma(\bar{v}) \subset TX.$$

- By above, we can put $\Gamma(\bar{v})$ inside nbhood of Δ in $X \times X$



- Because $\Gamma(\bar{v})$ is isotopic to 0-section in TX (by scaling $\bar{v} \to t\bar{v}$ $t \in [0,1]$),

its image in $X \times X$ is isotopic to Δ

- So $I(\Delta, \Delta)$ computed inside $X \times X$ equals

$$I(\Gamma(\bar{v}), \underset{\substack{\text{0-section,} \\ \text{diffeo to } X}}{X}) \quad \text{computed inside } TX$$

[Note: or indeed $I(X,X)$ computed inside TX]

- $\Gamma(\bar{v}) \pitchfork X$ iff the zeroes of $\bar{v}$ are <u>non-degenerate</u>, meaning: $d\bar{v}_x : T_x X \to T_x X$ is an iso, $\forall x : \bar{v}(x) = 0$

For such vector fields, one can prove by hand that

$\mathrm{ind}_x v = \pm 1 =$ sign of the corresp. intersection $\Gamma(f) \pitchfork X$

which proves $L(f) = \sum \mathrm{ind}_x v$

- For degenerate $\bar{v}$, need to perturb to $\tilde{v}$ w. nondegen zeros:

$$\bar{v} \rightsquigarrow \tilde{v}$$

degenerate zero — a bunch of non-degen zeros

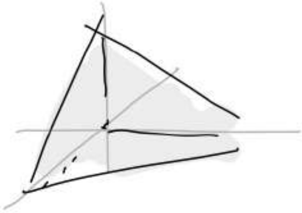
& prove that $\mathrm{ind}_x \bar{v} = \sum \mathrm{ind}_{x_i} \tilde{v}$.

Euler characteristic & triangulations

Def The std n-simplex is the polyhedron

$$\mathrm{Conv}\Big(\underbrace{\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}\begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix},\begin{pmatrix}0\\\vdots\\0\end{pmatrix}}_{n+1 \text{ pts in } \mathbb{R}^n}\Big) \subset \mathbb{R}^n$$

triangle



tetrahedron

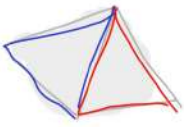
...

Def A triangulation of a mfd X^n is finite combinatorial data which prescribes a way of gluing a disjoint union of

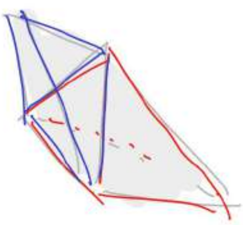
N n-simplices (for some N)

nicely together

[nicely means: the only gluing we allow is to glue together two faces of different simplices



dim 2



dim 3]

So that the resulting topo. mfd after gluing is homeo' to X.

<u>Note</u> One can demand more: that all simplices are actually embedded in X, and all faces are smooth submfds (except that we allow corners for the boundary)

<u>Theorem</u> Any mfd X admits a triangulation (in the stronger sense).

<u>Theorem</u> Let X be a mfd with a triangulation having the total of e_i faces of dimension i.

Then $\chi(X) = \sum (-1)^i e_i \quad \in \mathbb{Z}$

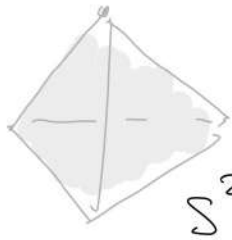
Example

S^1

$\chi = 3-3=0$

D^2

$\chi = 3-3+1=1$



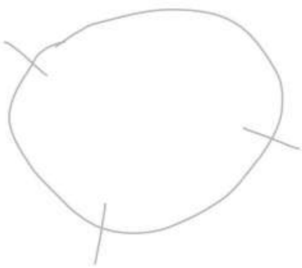
S^2

$\chi = 4-6+4=2$

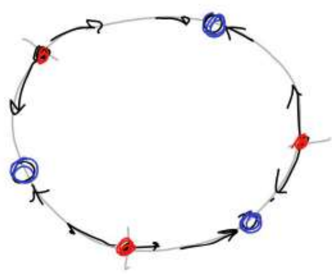
Idea One can construct a vector field $\bar{v}$ on X such that

- zeroes of $\bar{v}$ $\longleftrightarrow$ Barycentres of the faces e_i of all dimensions
- indices of $\bar{v}$ $\longleftrightarrow$ $(-1)^i$

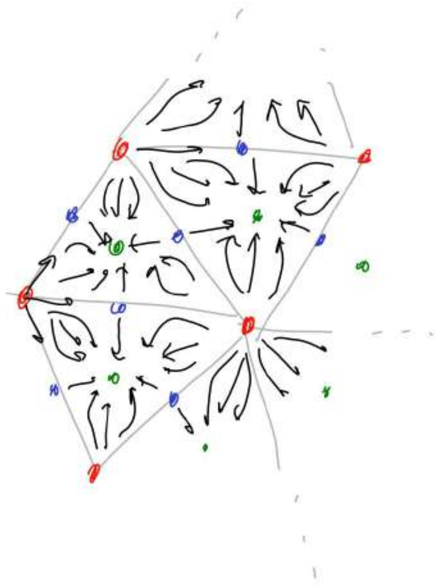
Here's how it works:



triangulation of S^1



$\bar{v}$



A 2-dim'l example locally