<u>Lec 18</u> Last time:

- Induced orientations on $f^{-1}(Z)$
- Oriented intersec thy & $\mathbb{Z}$-valued degree.

<u>Prop</u> $f: X \to Y$ smooth, $X = \partial W$, f extends to W

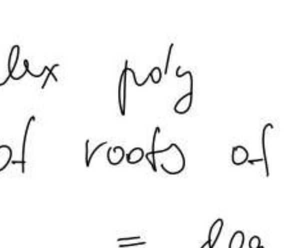
(X, Y oriented)

$\Rightarrow \deg f = 0$

Follows from more general thm about $I(f, Z)$ (earlier) taking $Z = pt$ $\square$

<u>Application of degree to finding roots of cx polys:</u>

<u>Prop</u> Let $W \subset \mathbb{C}$ comp. domain,



$P(z)$ complex poly [Note: can generalize to holo^c func].

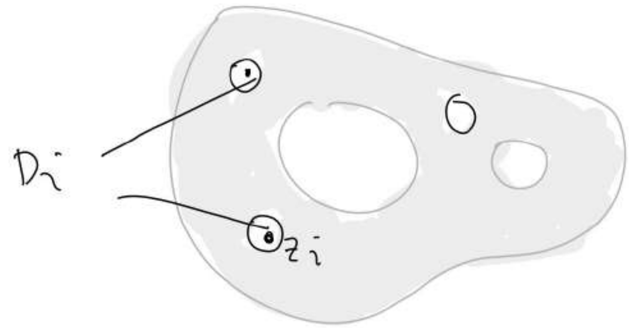
Then # of roots of P in W with multiplicities

$$= \deg\left(\frac{P}{|P|} : \partial W \to S^1\right)$$

<u>Proof</u> Let $z_i \in W$ be roots of P on W,

and $D_i \ni z_i$ be ε-disks around them

Set $W' = W \setminus (\sqcup D_i)$



$\frac{P}{|P|}$ extends to W' $\Rightarrow$ $\deg\left(\frac{P}{|P|} : \partial W \to S^1\right) =$

$$= \deg\left(\frac{P}{|P|} : \sqcup \partial D_i \to S^1\right)$$

by the above proposition, using $\partial W' = \partial W \sqcup (\overline{\partial D_i})$.
The rest follows from the claim: $\square$

<u>Claim</u> $\deg\left(\frac{P}{|P|} : \partial D_i \to S^1\right) =$ multiplicity of the root z_i.

<u>Proof</u> Let k be the order of z_i, then define $Q(z) = P(z) / (z - z_i)^k$.

Note: Q is non-vanishing on D_i.

Now take htpy

$$h_t(z) = \frac{z^k\, Q(z_i + trz)}{|Q(z_i + trz)|}$$

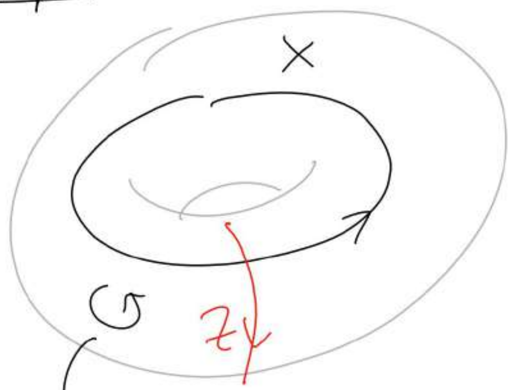
r is the radius of D_i (fixed)

is a htpy btw. $\text{const} \cdot z^\ell$ and $\frac{P(z_i + rz)}{|\quad - \| - \quad|} : S^1 \to S^1$

where the latter map is just a shift + rescaling of $\frac{P}{|P|} : \partial D_i \to S^1$. $\square$

<u>Recall</u> Intersection thy $\rightsquigarrow$ intersec numbers $I(X, Z)$ for two submfds $X, Z \subset Y$, $\dim X + \dim Z = \dim Y$.
Assuming X, Y, Z oriented, have $\underline{I(X, Z) \in \mathbb{Z}}$

<u>Example</u>



orientation on $Y = T^2$

We have: $I(X, Z) = -1$

Because: $\xrightarrow{1}$, $\downarrow 2$ is oppositely oriented as $\circlearrowleft$ (meaning: ${}^{2}\!\uparrow\!\xrightarrow[1]{}$)

<u>But</u> $I(Z, X) = +1$.

<u>More generally</u> $I(X,Z) = (-1)^{\dim X \cdot \dim Z} I(Z,X)$

<u>Proof</u> let $p \in X \pitchfork Z$ transverse intersec pt.
let β_1 be a pos. or. basis for T_pX,
β_2 — " — T_pZ.

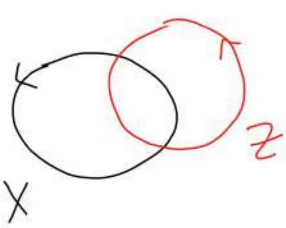
To compute the sign ± 1 with which p contributes to $I(X,Z)$, we concatenate:

(β_1, β_2)
& see if it is a pos orient basis for T_pY.

To compute how p contributes to $I(Z,X)$, we instead concatenate:
(β_2, β_1)
& compare orientation with Y.

But: (β_1, β_2) differs from (β_2, β_1)

by $\dim X \cdot \dim Z$ transpositions. $\square$

<u>Example</u> 

$Y = \mathbb{R}^2$. Here $I(X,Z) = 0$
(no matter how we orient.)

Def Let X be a compact, oriented mfd. Its Euler characteristic

$$\chi(X) \in \mathbb{Z}$$

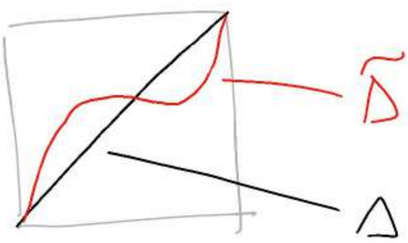
is defined to be

$$\chi(X) = I(\Delta, \Delta),$$

the self-intersection number of Δ with itself inside $X \times X$.

Note To compute $I(\Delta, \Delta)$, we need to perturb Δ to get $\tilde{\Delta} \subset X \times X$ isotopic to Δ and $\tilde{\Delta} \pitchfork \Delta$ & compute $I(\Delta, \tilde{\Delta})$

(which now makes sense by transversality)



Claim $\chi(X)$ does not depend on the choice of orient. of X.

Proof

Changing orient on X $\rightsquigarrow$ changes orient on Δ, $\tilde{\Delta}$, $X \times X$ such that all signs of intersections remain the same. $\square$

Prop If X is orientable odd-dim'l, then $\chi(X,X) = 0$

Proof $I(\Delta, \Delta) = (-1)^{\dim \Delta \cdot \dim \Delta} I(\Delta, \Delta) =$

$= -I(\Delta, \Delta) \Rightarrow 2I(\Delta, \Delta) = 0$ $\square$

Lefschetz Fixed point thm

Def Let $f : X \to X$ be a map. Its graph is:

$$\Gamma(f) = \{(x, f(x)) \; ; \; x \in X\} \subset X \times X$$

Def The Lefschetz number of f is:

$$L(f) = I(\Delta, \Gamma(f)) \in \mathbb{Z}$$

Lefschetz fixed point thm If $L(f) \neq 0$, f must have a fixed point.

Recall Fixed point $x \in X$ is s.that $f(x) = x$

Proof Fixed pts of f $\xleftrightarrow{1-1}$ $\Delta \cap \Gamma(f)$

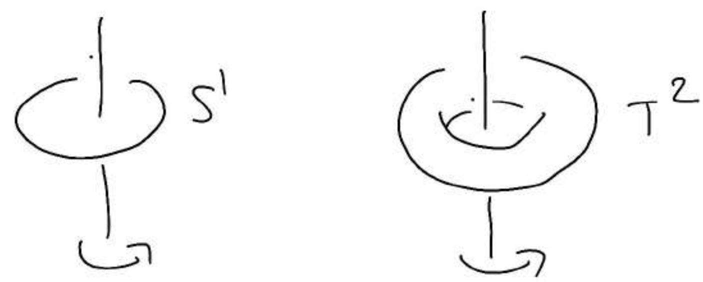
So if $\Delta \cap \Gamma(f) = \emptyset$, $L(f) = 0$ $\square$

<u>Prop</u> $L(f)$ is a homotopy invariant of f.
(Follows from general htopy inv'ce of intersec. numbers). $\square$

<u>Example</u> $L(\mathrm{Id}) = \chi(X)$, by definition.

<u>Corollary</u> If X has a smooth map $f: X \to X$ which is homotopic to Id & has no fixed pts, then $\chi(X) = 0$.

<u>Example</u> S^1, T^2 have such maps; eg rotation:



Note: $\chi(S^1) = 0$ because dim S^1 is odd,
but the fact that $\chi(T^2) = 0$ is not automatic.

<u>Def</u> $f: X \to X$ is called a Lefschetz map if
$\Gamma(f) \pitchfork \Delta$ — the intersec. is transverse

<u>Prop</u> Every map $f: X \to X$ is homotopic / can be perturbed to a Lefschetz map

<u>Proof</u> Use the way how we proved that a map can be perturbed to become $\pitchfork$ to a submfd:

$\exists$ ball S of some dimension
& smooth $F: X \times S \to X$ s.t.:

(a) $F(x,0) = f(x)$ &

(b) $s \mapsto F(x,s)$ is a submersion $S \to X$, $\forall x \in X$.

Now note: $G: X \times S \to X \times X$ defined by:

$$G(x,s) = (x, F(x,s))$$ is also a submersion.

$$dG = \left|\begin{array}{c|c} \mathrm{Id} & \text{"}\partial F/\partial x\text{"} \\ \hline 0 & \text{"}\partial F/\partial s\text{"} \end{array}\right|$$

this has full rank by (b).

Because G submersion, we certainly have $G \pitchfork \Delta$.

By Transversality thm:

for almost all s, the map $X \to X \times X$ defined by:

$$x \longmapsto G(x,s) = (x, F(x,s)) \text{ is } \pitchfork \Delta$$

<u>So</u> $x \longmapsto F(x,s)$ is Lefschetz (with s fixed.) $\square$

<u>Lemma</u> f is Lefschetz iff $\forall x \in \mathrm{Fix}\, f$,

df_x does not have eigenvalue $+1$

<u>Proof</u> $T(\Gamma(f)) = \begin{bmatrix} \mathrm{Id} \\ df \end{bmatrix} \begin{matrix} n \times n \\ n \times n \end{matrix}$ $\qquad n = \dim X$

$T(\Delta) = \begin{bmatrix} \mathrm{Id} \\ \mathrm{Id} \end{bmatrix} \begin{matrix} n \times n \\ n \times n \end{matrix}$

We have: $\begin{bmatrix} \mathrm{Id} & \mathrm{Id} \\ \mathrm{Id} & df \end{bmatrix}$ $(2n) \times (2n)$

has full rank (<u>ie</u> $\Gamma(f) \pitchfork \Delta$) iff df has no $+1$ eigenval. $\square$

<u>Def</u> If f is Lefschetz and $x \in \mathrm{Fix}\, f$, the <u>sign</u> of x

$$L_x(f) = \pm 1$$

is the sign of the corresponding intersec pt $\Gamma(f) \pitchfork \Delta$.

Then $$\boxed{L(f) = \sum_{x \in \mathrm{Fix}\, f} L_x(f)}$$

(by definition).

Prop The sign of any fixed point equals:

$$L_x(f) = \operatorname{Sign} \det (df_x - Id).$$

Proof Let $A = df_x$ and $\beta = (v_1 \dots v_n)$ a positive basis for $T_x X$. Then the following are positive bases:

for $T_{(x,x)} \Delta$: $\big((v_1, v_1), \dots, (v_n, v_n)\big)$

for $T_{(x,x)} \Gamma(f)$: $\big((v_1, Av_1), \dots, (v_n, Av_n)\big)$

Combine them to get a basis for $T_{(x,x)} X \times X$ & subtract as shown:

$$\big((v_1, v_1), \dots, (v_n, v_n), (v_1, Av_1), \dots, (v_n, Av_n)\big)$$

subtract

This doesn't change orientation & we get:

$$\big((v_1, v_1), \dots, (v_n, v_n), (0, (A-\mathrm{Id})v_1), \dots, (0, (A-\mathrm{Id})v_n)\big)$$

now subtract this way

& get:

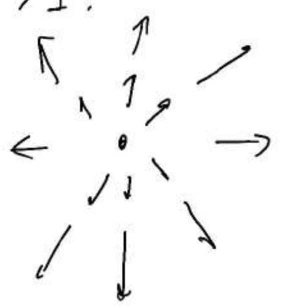
$$\big((v_1, 0), \dots, (v_n, 0), (0, (A-\mathrm{Id})v_1), \dots, (0, (A-\mathrm{Id})v_n)\big)$$

Clearly, the orientation of this basis is $\det(A-\mathrm{Id})$.

<u>Examples</u> ① Will look at linear maps $A: \mathbb{R}^2 \to \mathbb{R}^2$.
They have unique fixed point $0 \in \mathbb{R}^2$.

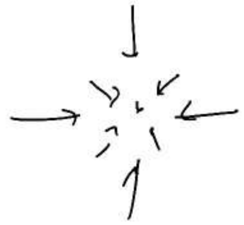
$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \qquad L_0(A) = \operatorname{sign}(a_1 - 1)(a_2 - 1)$$

- $a_1, a_2 > 1$:

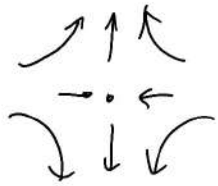


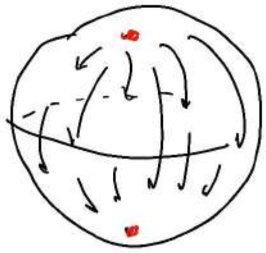
$L_0 = +1$ "Source"

- $a_1, a_2 < 1$:

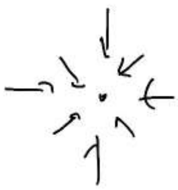
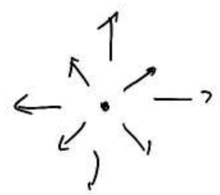


$L_0 = +1$ "Sink"

∘ $a_1 < 1 < a_2$  $L_0 = -1$ "saddle"

<u>Example</u>  "Time-1 flow of this vector field"

This map $S^2 \xrightarrow{f} S^2$ has 2 fixed pts looking like:

 and 

source sink

<u>So</u> $L(f) = +1+1 = 2$

Note: f is clearly homotopic to Id
[all time-1 flows of vector fields are — by
taking time-t flow, $t \in [0,1]$]

<u>So</u> $\chi(S^2) = +2$.

Orientable surfaces:

sphere "genus $g=0$" — torus "genus $g=1$" — "genus 2" — "genus g" $g \geq 0$

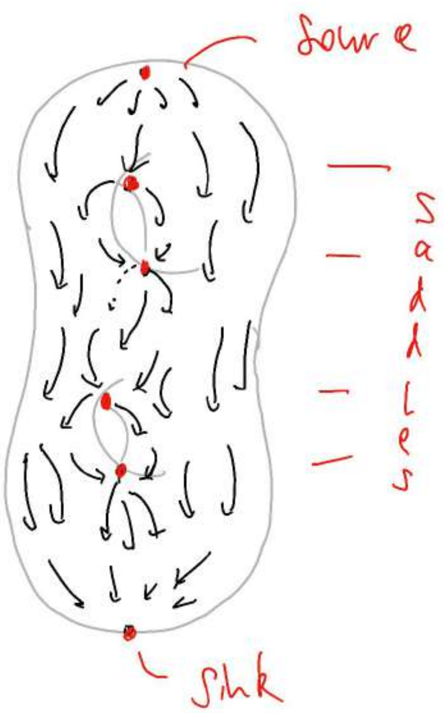
Classification thm for surfaces

$\forall$ closed orientable 2-diml mfd is diffeomorphic to a genus g surface, for some $g \geq 0$.

<u>Prop</u> Let Σ_g be the genus g surface. Then

$$\chi(\Sigma_g) = 2 - 2g$$

<u>Proof</u>



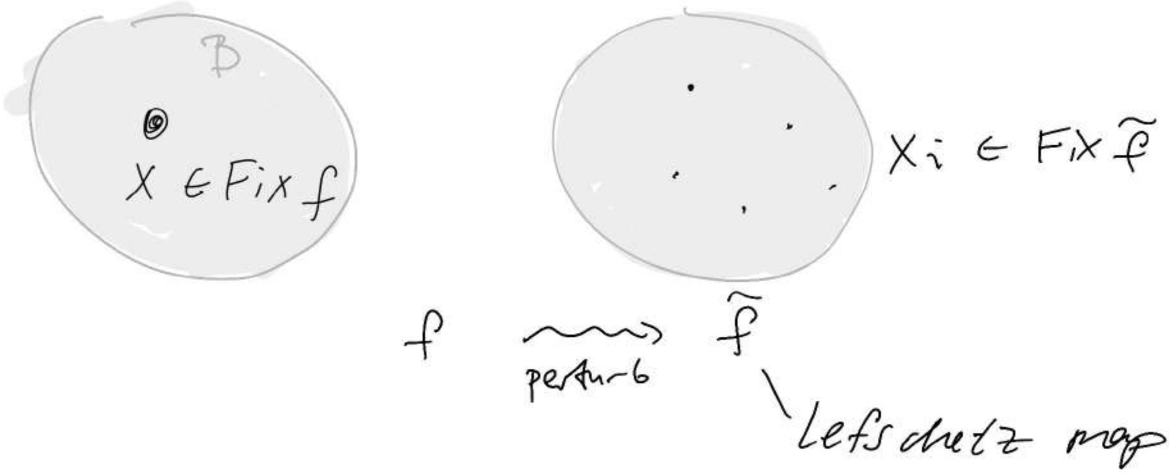
Σ_g has a Lefschetz map with
1 source, 1 sink & $2g$ saddle points

& that map is htopic to Id

<u>So</u> $\chi(\Sigma_g) = L(\mathrm{Id}) = 1 + 1 - 2g$ $\square$

What if $f: X \to X$ has isolated fixed points, but is not Lefschetz?

After a perturbation get:



<u>Prop</u>

$$\sum L_{x_i}(\tilde{f}) = \deg\left(\frac{f(x)-x}{|f(x)-x|} : \partial B \to S^{n-1}\right)$$

these are ± 1's

Proof is omitted (can take as an exercise).