Lec 17

Claim Consider an orientation on $X \times [0,1]$.
By above, it induces an orientation on

$$\partial (X \times [0,1]) = X \times \{0\} \sqcup X \times \{1\}.$$

The induced orientations on $X \times \{0\}$ and $X \times \{1\}$ are opposite, wrt the obvious identification $X \times \{0\} \cong X \times \{1\}$.

Proof

$T_x(X \times [0,1]) = T_x X \times \mathbb{R}$;
for $x \in X \times \{0\}$, the inward pointing vector is

$$\begin{array}{cc} (0, & 1) \\ \cap & \\ T_xX & \mathbb{R} \end{array}$$

But for $x \in X \times \{1\}$, the inward pointing vector is

$$(0, -1).$$

For a basis β of $T_x(\partial X)$, $\beta \cup (0,1)$ is pos. orient on X

$\iff \beta \cup (0,-1)$ is neg. orient on X

so $X \times \{0\}$ & $X \times \{1\}$ have opposite induced orientations $\square$

<u>Notation</u> $\partial(X \times [0,1]) = X_1 \cup \overline{X}_0 = X_1 - X_0$

where $X_i = X \times \{i\}$, $i = \{0,1\}$.

<u>The 1-dim'l case</u> If X is an oriented 1-mfd, then the induced orientation on ∂X is defined by the following rule:

Let $p \in \partial X$ and $v \in T_p X$ inward pointing; if $\{v\}$ is pos. oriented then put $\oplus$;

otherwise put $\ominus$.

orientation induced orientation

<u>Lemma</u> If X is an oriented <u>compact</u> 1-mfd w. bdy, then the <u>signed</u> count of the points in ∂X, using the induced orientation, is <u>zero</u>

Proof Obvious (by classification of 1-mfds). $\square$

Orientation on preimages

Lemma Consider a direct sum of v.spaces:

$$V = V_1 \oplus V_2.$$

The choice of orientation on any two of these spaces induces a canonical orientation on the third one

Proof

Note: two ordered bases β_1 for V_1, β_2 for V_2 give an ordered basis (β_1, β_2) for V.

(a) Assume V_1, V oriented then
choose β_1 pos. oriented & take β_2 (a basis for V_2) s.t. (β_1, β_2) is pos. oriented on V.
The orientation class of β_2 defines an orient. on V_2

(b) Assume V_1, V_2 oriented, then
choose β_1, β_2 pos. oriented & define the orient. of V by the orientation class of (β_1, β_2).

$\square$

Note when V_1, V_2 are oriented, the induced orientations on $V_1 \oplus V_2$ and $V_2 \oplus V_1$ differ by the sign $(-1)^{\dim V_1 \dim V_2}$

Because the bases (β_1, β_2)

dim V_1 vectors dim V_2 vectors

and

(β_2, β_1)

differ by $\dim V_1 \cdot \dim V_2$ permutations, and every permutation of a basis:

$$(v_1 \dots v_i\, v_{i+1} \dots v_k) \longrightarrow (v_1 \dots v_{i+1}\, v_i \dots v_k)$$

changes the orientation class (seen last time: the corresp. matrix has det $= -1$).

Now let $f: X \to Y$ smooth map,

$Z \subset Y$ submfd,

$f \pitchfork Z$, $\partial f \pitchfork Z$, so that

$f^{-1}(Z) \subset X$ is a mfd with bdy.

Proposition Orientations on X, Y and Z canonically induce an orientation on $f^{-1}(Z)$.

(In particular, $f^{-1}(Z)$ is then orientable).

Proof Suppose $x \in X$, $f(x) = z \in Z$. Denote:

$$S = f^{-1}(Z).$$

Let $N_x(S;X) = (T_x S)^{\perp}$ - orthog. complement taken in $T_x X$.

Then:

$$N_x(S;X) \oplus T_x(S) = T_x(X).$$

Need to show Orients on Y, Z give an orient. on $N_x(S;X)$. Then an orient. on S is induced by the direct sum above

By defn of $f \pitchfork Z$:

$$df_x(T_x(X)) + T_z(Z) = T_z Y$$

may intersect!

But also: $df_x^{-1}(T_z Z) = T_x S$

full preimage of subspace

so have direct sum:

$$df_x\,(N_x(S;X)) \oplus T_z Z = T_z Y$$

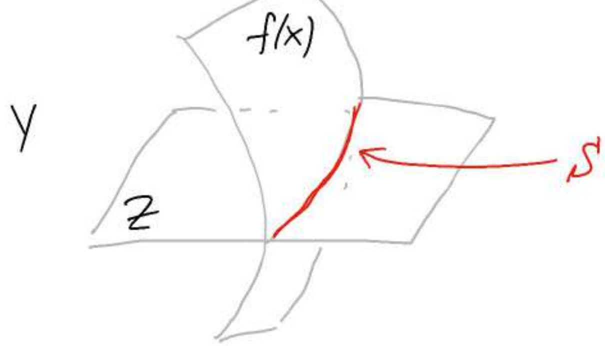
now these don't intersect.

So indeed, $N_x(S;X)$ has induced orientation from Y & Z by direct sum.
Smoothness of orientation: easy to check (all splittings vary smoothly with x) $\square$

<u>Note</u> Instead of $N_x(S;X)$, can choose any $H \subset T_x X$ complementary to $T_x S$ & orient S by direct sum orientations via:

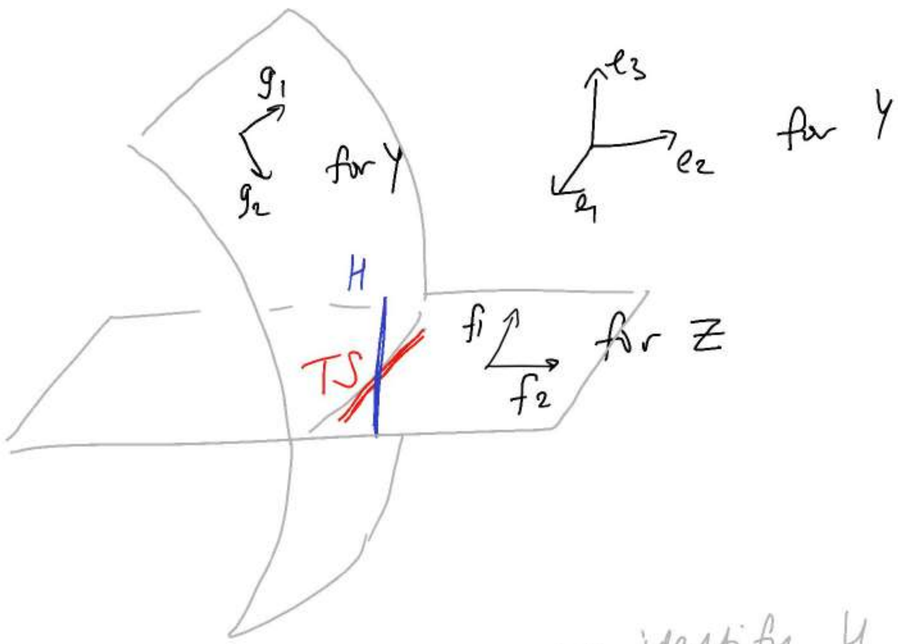
$$\boxed{\begin{array}{c} df_x(H) \oplus T_z(Z) = T_z(Y) \\ H \oplus T_x(S) = T_x(X) \end{array}}$$

<u>Example</u> $Y = \mathbb{R}^3$, $Z = \mathbb{R}^2 \subset \mathbb{R}^3$, $X = \mathbb{R}^2$, $f: X \to Y$ an embedding shown.



Then S can be identified with $f(S)$, a curve.

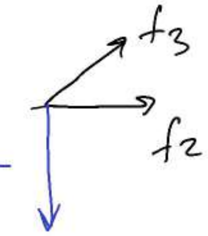
Fix orientations on Y, Z, $f(X)$:



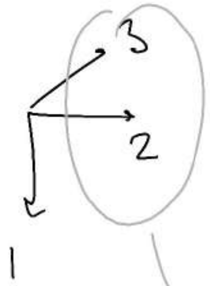
we identify H with $Df(H)$

ⓐ Orient H via: $H \oplus TZ = TY$;

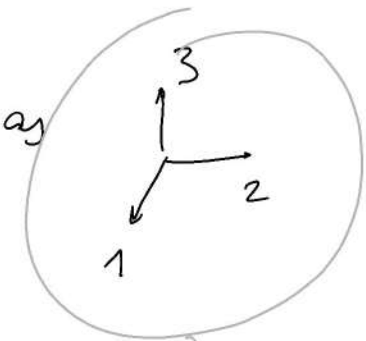
Orientation on H



The orientation is like this because :



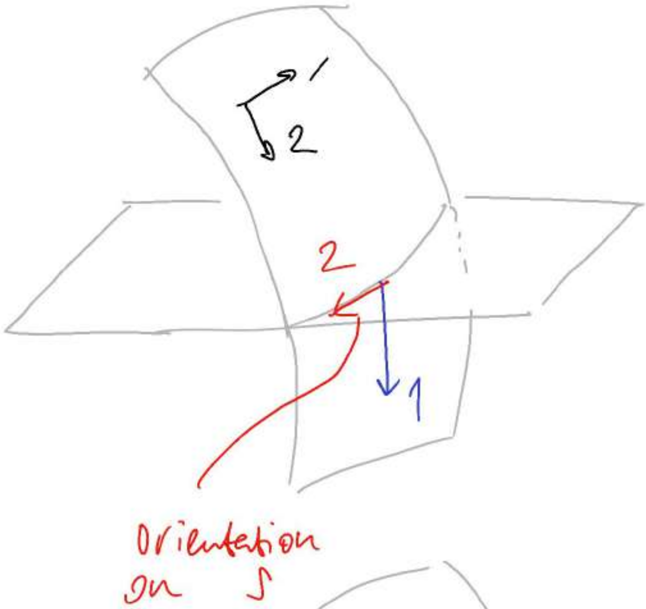
is oriented same as



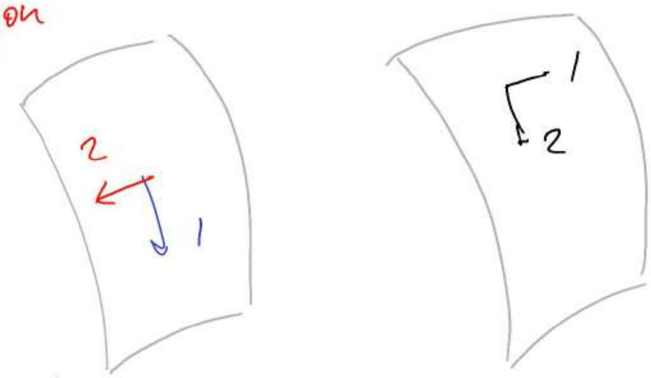
come from TZ

comes from TY.

(b) Orient TS via: $H \oplus TS = TX$:

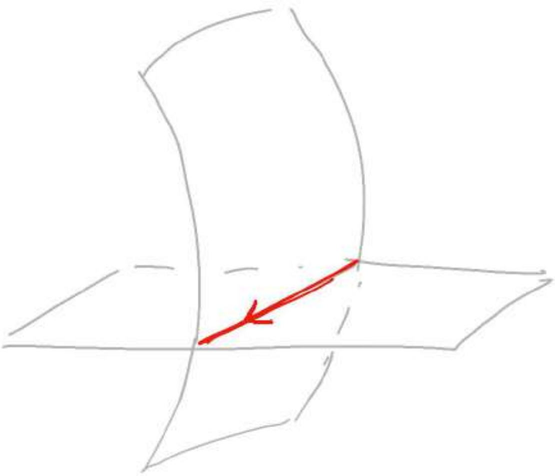


because:



are oriented the same.

<u>Answer</u> S is oriented this way:



What happens at $\partial f^{-1}(Z)$?

Observe: If X, Y, Z oriented, can orient $\partial f^{-1}(Z)$ in two ways:

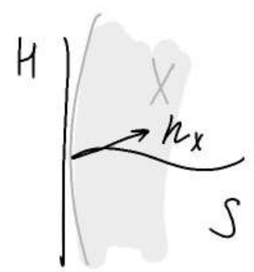
ⓐ – take preimage orient. on $f^{-1}(Z)$,
– then bdy orient. on $\partial f^{-1}(Z)$

ⓑ – take bdy orient. on ∂X
– consider $\partial f = f|_{\partial X}$, and take preimage orient. on $(\partial f)^{-1}(Z) = \partial f^{-1}(Z)$.

Prop These two orientations differ by $(-1)^{\operatorname{codim} Z}$.

Proof let $S = f^{-1}(Z)$, take $H \subset T_x(\partial X)$ st
$$H \oplus T_x(\partial S) = T_x(\partial X)$$

Note Also $H \oplus T_x(S) = T_x(X)$



Let n_x be the inward normal.

Method ⓑ: $\underline{T_x(\partial X)} \oplus \langle n_x \rangle = T_x X$

orient this

then: $H \oplus \underline{T_x(\partial S)} = T_x(\partial X)$

orient this

(assuming H has been oriented via f, Y, Z.)

Method ⓐ: $H \oplus \underline{T_x S} = T_x X \quad \leftrightarrow T_x \oplus S$

orient this

then: $\underline{T_x(\partial S)} \oplus \langle n_x \rangle = T_x S$

orient this

Exercise complete the proof □

$\boxed{\text{Oriented intersection thy}}$

Setup

X, Y, Z without bdy, X comp., $Z \subset Y$,
$\dim X + \dim Z = \dim Y$

Additionally X, Y, Z oriented.

If $f: X \to Y$, $f \pitchfork Z$, then $f^{-1}(Z)$ is a finite # of pts.

Def let $p \in f^{-1}(Z)$; the orientation number of x is ± 1 depending on the following rule.

We have $df_x(T_x X) \oplus T_z Z = T_z Y$ $\quad (\star)$

by transversality (and $\dim X + \dim Z = \dim Y$)
& $df_x : T_x X \to df_x(T_x X)$ is an iso
$\Rightarrow$ orient. of X gives orient of $df_x(T_x X)$

We say the sign is $+1$ if pos. oriented bases β_1, β_2 for $df_x(T_x X)$, $T_z Z$

give the basis (β_1, β_2) which is pos. orient for Y.

need to fix order, otherwise all signs may change.

Def $$I(f, Z) = \sum_{x \in f^{-1}(Z)} \underbrace{\pm 1}_{\text{orientation number at } x}$$

Prop If $X = \partial W$ and $f : X \to Y$ extends to W, then $I(f, Z) = 0$ (W compact).

Proof let $F: W \to Y$ be an extension of f.
By Extension thm: may assume $F \pitchfork Z$
(recall: given $f \pitchfork Z$).

$\Rightarrow$ $F^{-1}(Z)$ oriented 1-mfd w. bdy $\partial F^{-1}(Z) = f^{-1}(Z)$
$\Rightarrow$ $(\pm)$ $I(f, Z)$ is the signed count of
$\partial F^{-1}(Z)$ with orient. induced from $F^{-1}(Z)$
[compare previous prop.]

but $\# \partial F^{-1}(Z) = 0$ by the beginning of this lecture $\square$

Prop If $f_0 \underset{\text{htopic}}{\sim} f_1$ & $f_0, f_1 \pitchfork Z$ then

$$I(f_0, Z) = I(f_1, Z)$$

Proof let $F: X \times [0,1] \to Y$ be a htpy, then

$$I(\partial F, Z) = 0 \text{ by previous Prop,}$$

where $\partial F = F|_{X_0 \sqcup \overline{X_1}}$ opposite orient

Recall: $X_0 = X \times \{0\}$ & $X_1 = X \times \{1\}$ have
opposite orientations induced from $\partial (X \times [0,1])$

0 — because $F^{-1}(z)$ 1-dim'l mfd

$$\underline{\text{So}} \quad \overset{\overset{0}{\|}}{\# \partial F^{-1}(z)} = f_1^{-1}(z) - f_0^{-1}(z) =$$

signed count of bdy pts with induced orientation

signed count of pts with orientation signs

$$= I(f_1, z) - I(f_0, z) \qquad \square$$

<u>Def</u> When f is not necessarily $\pitchfork Z$, let g be any map htopic to f s.t. $g \pitchfork Z$, and define

$$I(f, Z) \underset{\text{def}}{=} I(g, Z).$$

By previous prop, this is well-def (does not dep. on choice of g)

<u>Particular case</u> ($\mathbb{Z}$-valued) degree of

$$f: X^n \to Y^n$$

defined by: $\deg f = I(f, pt)$

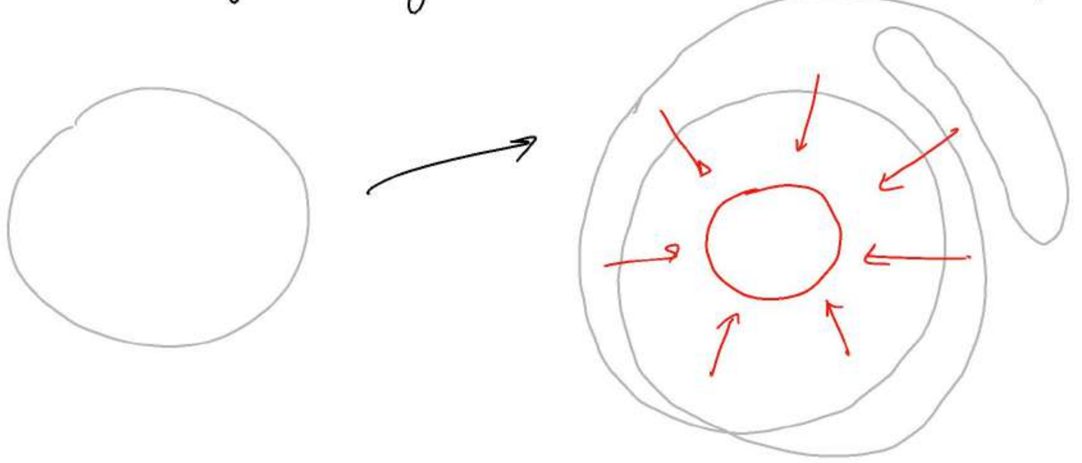
Alternatively: take $x \in f^{-1}(pt)$ regular & attach signs depending on whether

$$df: T_x X \to T_{f(x)} Y$$

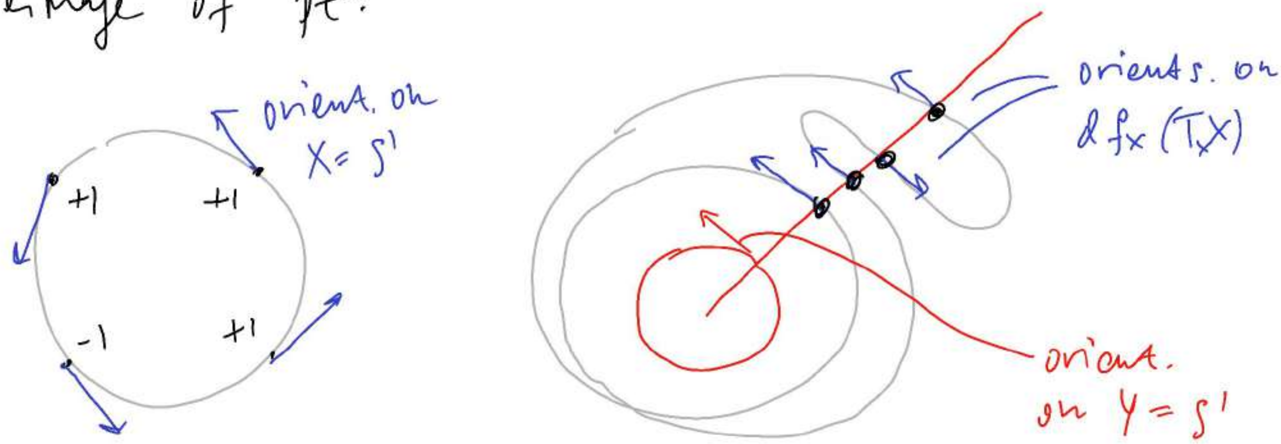
preserves orientation & sum $x \in f^{-1}(pt)$ with signs.

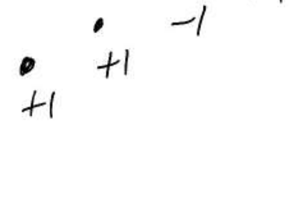
Example

$S^1 \to S^1$ given by the shown wrapping + projection to $\bigcirc$



Preimage of pt:





$\deg f = 1+1-1+1 = 2.$