Lec 16 Recall last time:

- for $f: X \to Y$ a map,
  $Z \subset Y$ submfd,

  $\dim X + \dim Z = \dim Y$,

  defined the mod 2 intersec number
  $$I_2(f, Z) \in \{0, 1\},$$
  assuming $f \pitchfork Z$.
- using the transversality techniques we developed, we proved that $I_2(f, Z)$ is invt under htopies of f.
- in particular, have $I_2(X, Z)$ if $X, Z \subset Y$ are submfds of complem. dim; did some examples.
- defined winding number and sketched that it computes the # of roots of a fction inside a given domain

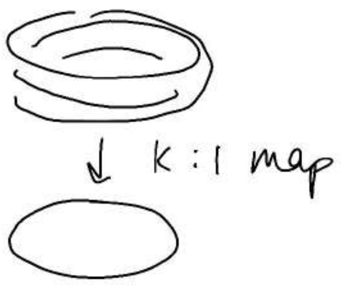
Next goal Oriented intersection thy.

Why? (One of) the aims of intersec thy is to distinguish non-htopic maps, and intersec. thy mod 2 is sometimes insufficient.

Eg $S^1 \xrightarrow{f_k} S^1$

$z \longmapsto z^k$

$S^1 = \{z \in \mathbb{C} \ \ |z| = 1\}$



then $\deg_2 f_k = I(f, \{pt\}) = k \mod 2$.
although f_k and f_l are not htopic when $k \neq l$, $\deg_2$ does not distinguish all of them!

Solution Define $\mathbb{Z}$-valued degree, $\deg f \in \mathbb{Z}$; we will then have

$$\deg f_k = k$$

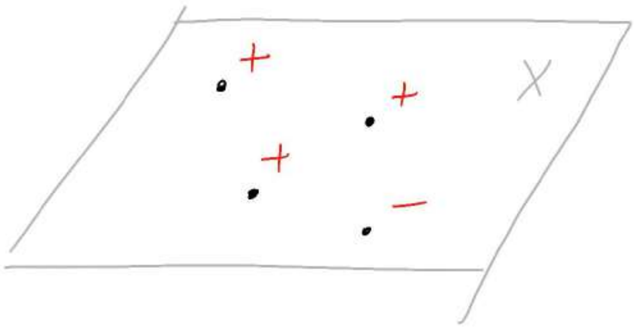
which distinguishes all the f_k up to htopy.

Idea for the degree of $f: X^n \to Y^n$.

ⓐ Introduce notion of orientation on a manifold.
Assume X, Y oriented.

(b) Taking $p \in Y$ regular point for f,
recall that $f^{-1}(p)$ is a finite # of pts.

To each point $x \in f^{-1}(p)$, assign a sign: $+$ or $-$



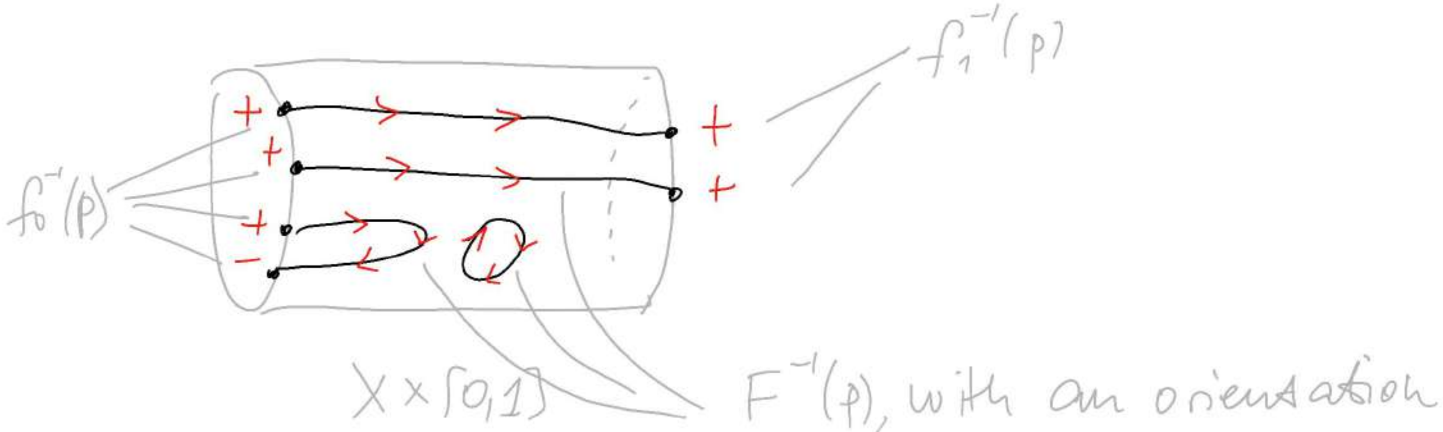
(c) Count the points in $f^{-1}(p)$ with signs,
eg in the above picture: $+1+1+1-1 = 2 \in \mathbb{Z}$.
Define the degree
$$\deg f = \text{the above signed sum}$$

(d) To prove that $\deg f$ is htopy invt,
prove that $\forall Z \subset Y$ submfd s.t. $f \pitchfork Z$,

the submanifold $f^{-1}(Z)$ carries a canonical orientation

In particular, if $F : X \times [0,1] \to Y$ is a
homotopy between $f_0, f_1 : X \to Y$,

$F^{-1}(p)$ is an oriented 1-mfd with bdy



Then $F^{-1}(p)$ is a disjoint union of closed intervals $\bigsqcup_i I_i$, plus circles.

Using orientation they can show:

- If an interval $I_i \subset F^{-1}(p)$ connects the same boundary compt. of $X \times \{0,1\}$, ie $X \times \{0\}$ or $X \times \{1\}$, then

  $\partial I_i \subseteq f_0^{-1}(p)$ resp. $f_1^{-1}(p)$

  is a pair of points having opposite signs wrt. f_0 / f_1.

- If an interval $I_i \subset F^{-1}(p)$ connects the opposite bdy compts. of $X \times \{0,1\}$, ie

  ∂I_i has one pt $\in X \times \{0\}$

  and another pt $\in X \times \{1\}$

  then these two points have the same sign wrt. f_0 resp. f_1.

  (see picture above)

We will now develop orientation thy

Def let V be a finite-dim real vector space,

$\beta = \{v_1 \ldots v_n\}$,

$\beta' = \{v_1', \ldots, v_n'\}$ two ordered bases of V.

Then $\exists!$ linear iso^m $A : V \to V$

such that $\beta' = A\beta$.

We say β, β' have the same orientation / are equivalently oriented

if $\det A > 0$,

and β, β' are oppositely oriented if $\det A < 0$

Lemma (If $\dim V \neq 0$) V has exactly 2 equivalence classes of equivalently oriented bases:

$$\left| \{\text{bases of } V\} / \text{equivalently oriented} \right| = 2$$

Proof let $\beta_1 = \{v_1, v_2, \ldots, v_n\}$ be any basis

and let $\beta_2 = \{-v_1, v_2, \ldots, v_n\}$

changed sign of the first vector.

Then $\forall$ basis β is either equiv. orient with β_1 or with β_2. Indeed, let

$$A\beta = \beta_1$$

If $\det A > 0$, we are OK, otherwise

$$A \cdot \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & \ddots & \\ & & & +1 \end{pmatrix} \beta = \beta_2$$

and $\det \left(A \cdot \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & \ddots & \\ & & & +1 \end{pmatrix} \right) = -\det A > 0$ $\square$

Def An orientation on V is a choice of one of the two equivalence classes of $\{$bases mod orientation equivalence$\}$.

Note There is no canonical choice if V is an abstract v. space and comes with no fixed basis. Of course, choosing a basis of V immediately defines an orientation (by taking the equiv. class of that basis).

Addition If $V = \{0\}$, an orientation of V is an abstract choice of a sign: $+$ or $-$.

Def Let V be an oriented v. space, and β a basis of V. We say β is positively resp. negatively oriented if it belongs / does not belong to the equiv. class defined by the orientation on V.

Def Let $A: V \to W$ be a linear isomorphism btw. oriented v. spaces. Pick β a pos. oriented basis of V. We say f preserves / reverses orientation if $f\beta$ is a pos/neg. oriented basis of W.

Def Let X be a mfd. An orientation on X is a smooth choice of orientations for all tangent spaces $T_x X$.

Here, smooth choice means:
$\forall x \in X^k$ $\exists$ a parametrisation:

$$U \underset{\text{open}}{\subseteq} \mathbb{R}^k, \quad \varphi: U \to X,$$
$$\varphi(U) \text{ contains } x$$

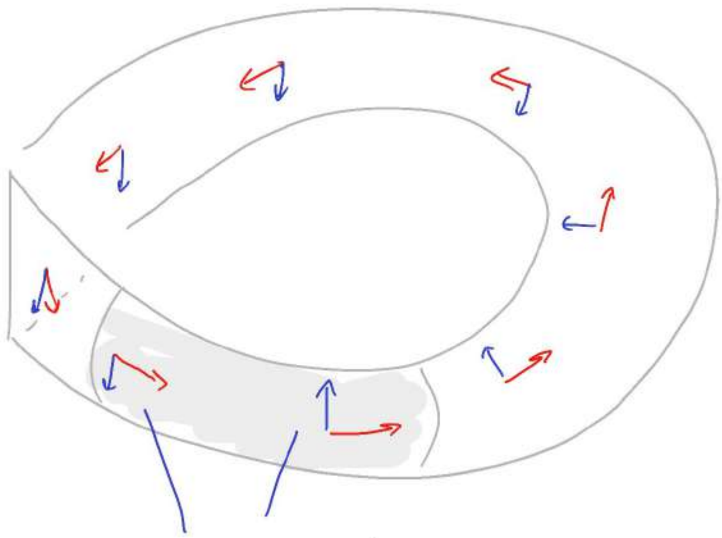
such that $d\varphi_u : \mathbb{R}^k \xrightarrow[\cong]{} T_{\varphi(u)} X$

is orientation-preserving w.r.t. the std orientation on $\mathbb{R}^k$ and the chosen orientation on $T_{\varphi(u)}X$.

Note X can have bdy; the defn still works.

Not all mflds are orientable!

Example Möbius strip.



opposite orientations in this chart!

Thm (without proof) $\forall$ simply-connected mfld is orientable.

I.e. If X is non-orientable, $\exists$ loop $\gamma \subset X$ s.t. "orientation gets reversed along γ" & γ has to be non-contractible.

<u>Prop</u> A <u>connected</u>, <u>orientable</u> mfd (maybe w. Bdy) X has exactly 2 orientations.

<u>Proof</u> Take two orientations on X; will prove: the set where they agree / disagree are both open. (Clearly, this proves Prop.)

let $x \in X$, and take parametrisations:

$$h : \underset{\text{open}}{U \subset \mathbb{R}^k} \longrightarrow X$$

$$\tilde{h} : \tilde{U} \longrightarrow X$$

$h(U), h(U') \ni x$
wlog. assume $h(0) = \tilde{h}(0)$.

such that dh preserves the first orient. on X, and $d\tilde{h}$ preserves the second one

The two orientations on $T_x X$ agree

$\Updownarrow$

$$d(h^{-1} \circ \tilde{h})_0 : \mathbb{R}^k \longrightarrow \mathbb{R}^k$$

preserves orientation <u>ie</u> has $\det > 0$.

<u>then</u> $d(h^{-1} \circ \tilde{h})$ has $\det > 0$ in an open nbhood of 0
<u>so</u> the orientations on X agree / disagree simultaneously at any point in nbhd of x. $\square$

Def An oriented mfd is a manifold with a chosen orientation.

Observe Oriented mfd $\rightsquigarrow$ orientation has been fixed.
Orientable mfd $\rightsquigarrow$ orientation can be chosen, but we have not fixed one

Lemma If X, Y are oriented then $X \times Y$ has a canonical orientation.

Proof
$$T_{(x,y)}(X \times Y) = \underset{\alpha}{T_x X} \times \underset{\beta}{T_y Y}$$
choose pos. oriented bases α, β

Then $\alpha \times 0 \cup 0 \times \beta$ is a basis for $T_x X \times T_y Y$, and it gives the desired orientation. $\square$

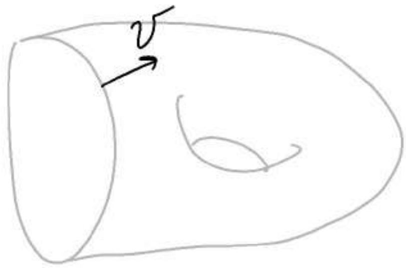
Lemma If X is oriented, then ∂X has a canonical orientation.

Proof Let $x \in \partial X$. Recall:
- $T_x(\partial X) \subset T_x X$ is codim 1
- $\exists$ uniquely defined unit vector $v \in T_x X$

such that $v \perp T_x(\partial X)$

and v "points inwards" into X.

(see prev. ex. sheet).



To define a (canonical) orientation on $T_x(\partial X)$, pick any basis β of $T_x(\partial X)$ such that

$\beta \cup \{v\}$ is a pos. oriented basis for T_xX (wrt the orientation on X). $\square$

Eg