Lec 14

Recall last time:

- finished $\deg_2$ of a map

- proved transversality thm:
  if $F : X \times S \to Y \pitchfork Z$ & $\partial F \pitchfork Z$
  then for almost all s,
  $f_s : X \to Y \pitchfork Z$

- defined ε-nbhoods Y^ε of Y

- proved that $\exists$ submersion $\pi : Y^\varepsilon \to Y$
  It can be constructed by taking
  $w \in Y^\varepsilon$ to <u>closest</u> pt in Y, but we
  provided a more abstract (& easier)
  construction.

$$\begin{array}{c} \mathbb{R}^N \\ \cup \end{array}$$

<u>Corollary</u> Let $f : X \to Y$ smooth, Y without bdy
Then $\exists$ open ball $S \subset \mathbb{R}^N$
& a map $F : X \times S \to Y$

st $F(x,0) = f(x)$ <u>and</u>

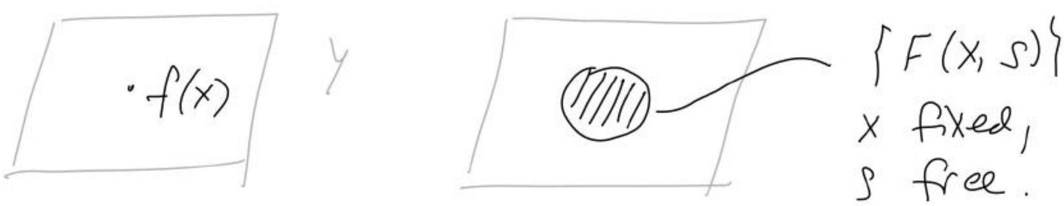
$\forall$ fixed $x \in X$, the map

$$s \longmapsto F(x,s)$$
$$S \longrightarrow Y$$

is a submersion.

<u>In particular</u> $F, \partial F$ are submersions.

<u>What it means</u> we deform f with $\dim S = N$ parameters s.t. the deformations at any point give a submersion $S \to Y$:



<u>Proof</u> Define:

$$F(x,s) = \pi\Big(\underbrace{f(x) + \varepsilon f(x) s}_{\in Y^{\varepsilon}}\Big)$$

$s \in \mathbb{R}^N$

submersion $Y^{\varepsilon} \to Y$

Note: $F(x,0) = f(x)$. For fixed x,

$x \mapsto f(x) + \varepsilon f(x) s$ is a submersion

as a map $S \to Y$. Because π is subm., $F(x,s)$ is a submersion for each fixed x. $\square$

Transversality htopy thm

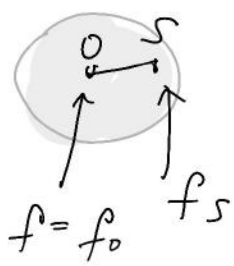
Let $f : X \to Y$ smooth, $Z \subset Y$ submfd
(Y, Z without bdy)

then $\exists\, g : X \to Y$ htopic to f
such that $g \pitchfork Z$, $\partial g \pitchfork Z$

this symbol is for transverse maps

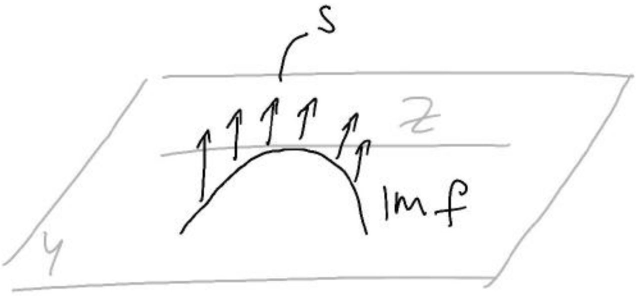
Proof Take $F : X \times S \to Y$ from prev. corollary, then by Transversality thm,
$f_s : X \to Y$ is $\pitchfork Z$

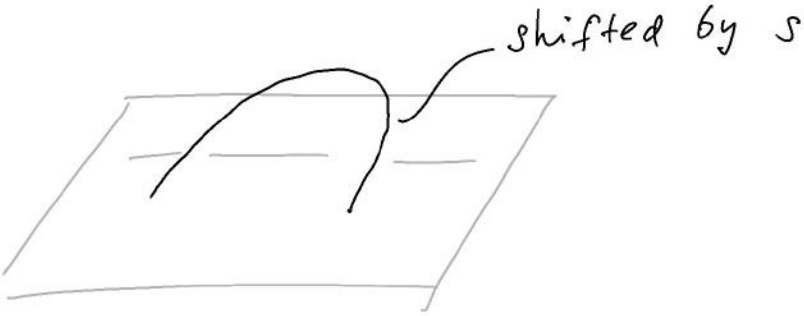
for almost all s. All f_s are htopic to f:
connect $s \in S = B^N$ with $0 \in S = B^N$
by some path. $\square$

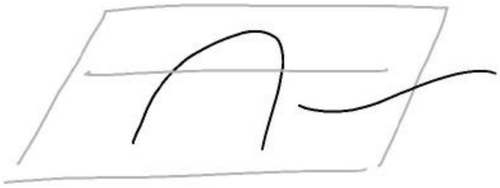


Quick look back at what we did:

Given $f : X \to Y$, deformed by shifting it by a generic vector εs (small) & projecting to Y







Im f_s: projected back
& became transversal.

[Needed crucially a submersion $Y^{\varepsilon} \to Y$]

A stronger statement:

<u>Extension thm</u> $Z \subset Y$ closed submfd,
$C \subset X$ closed subset.

Let $f: X \to Y$ be smooth and
$f \pitchfork Z$ over C, $\partial f \pitchfork Z$ over $C \cap \partial X$.

Then $\exists$ smooth $g: X \to Y$ htopic to f
& s.t. $g \pitchfork Z$, $\partial g \pitchfork Z$,
$$g|_C \equiv f|_C.$$

Meaning If f was transverse over C, can homotop to make it transverse everywhere leaving f unchanged on C.

Lemma If $\mathcal{U} \supset C$ open nbhood of a closed set C then $\exists$ smooth $\gamma: X \times [0,1]$ s.t.:

$$\gamma|_C \equiv 0$$
$$\gamma|_{X \setminus \mathcal{U}} \equiv 1$$

Proof Skip (partition of unity) $\square$

Proof of thm Observe: $f \pitchfork Z$ over C

$$\Downarrow$$

$f \pitchfork Z$ over $\mathcal{U} \supset C$ some nbhd

[Because transversality is an open condition: if $\operatorname{Im} d_x f + TZ = TY$ then the same holds for any x' in nbhood, if $f(x), f(x') \in Z$]

Let γ be from the lemma, $\tau = \gamma^2$, then $d\tau = 0$ on C.
Recall the map $F: X \times S \to Y$ from Htopy thm.

Modify: $G(x, s) := F(x, \tau(x) s)$

Claim $G \pitchfork Z$.

Proof Take $(x,s) \in G^{-1}(Z)$. Consider cases:

(a): $\tau(x) \neq 0$, then with fixed x, the map

$$r \longmapsto \underset{\substack{\| \\ f(x) + \underbrace{\varepsilon \tau(x) r}}}{G(x,r)} \quad \text{is a submersion}$$

depends on r linearly when $\tau(x) \neq 0$.

So whole G is also a submersion.

(b): then $dG = df + \varepsilon \underbrace{(d\tau)}_{0} \cdot r + \underbrace{\varepsilon \tau}_{0} \underbrace{(dr)}_{\text{Id}} = df$

or more precisely:

$$d G(\underset{TX}{u}, \underset{TS}{v}) = df(u)$$

Given that $f \pitchfork Z$ over $\{\text{points where } \tau = 0\} \subset U$ same holds for G.

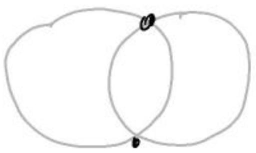
Analogously for ∂G. $\square$

Intersection thy mod 2

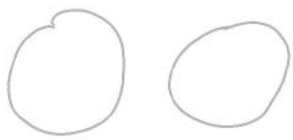
<u>Setup</u> $X, Z \subset Y$ closed mfds of <u>complementary dimension</u> : $\dim X + \dim Z = \dim Y$.

<u>Plan</u>:

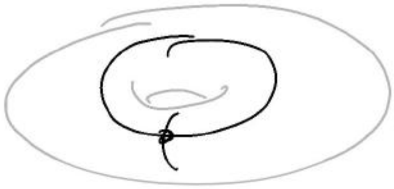
- Perturb X, Z so that they become $\pitchfork$
- Then $X \cap Z$ is a finite number of pts; count them mod 2 : $I_2(X, Z)$.
- The quantity is indep. of htopies of embedding $X, Z \subset Y$.



$I_2 = 2 = 0 \bmod 2$



$I_2 = 0$



$I_2 = 1$

A more convenient setup:

<u>Def</u> Assume $f : X \to Y$ is smooth,
$Z \subset Y$ submfd,
$f \pitchfork Z$,
$\dim X + \dim Z = \dim Y$.

Then (given X, Z closed), $f^{-1}(Z)$ is a finite #
of points. Define
$$I_2(f, Z) = \# f^{-1}(Z) \mod 2.$$

<u>Thm</u> If $f_0, f_1 : X \to Y$ both $\pitchfork Z$
and are htopic, then $I_2(f_0, Z) = I_2(f_1, Z)$

<u>Proof</u> Take a htopy $F : X \times [0,1] \to Y$.
By <u>Extension thm</u>: can assume $F \pitchfork Z$.

<u>So</u> $F^{-1}(Z) =$ 1-mfd w. Bdy
$\partial F^{-1}(Z) = f_0^{-1}(Z) \sqcup f_1^{-1}(Z)$ - even # of pts
□

<u>Def</u> If $f : X \to Y$ is not $\pitchfork Z$,
perturb it so it becomes $\pitchfork$ to define
$I_2(f, Z)$ (possible by Htopy Transversality thm)

<u>Cor</u> $I_2(f, Z)$ does not depend on
homotopies of f. □

<u>Boundary thm</u> Suppose $X = \partial W$
and $g : X \to Y$ smooth. If g extends to W
then $I_2(g, Z) = 0$ $\forall Z \subset Y$ of complementary
dim. to X.

Proof Let $G: W \to Y$ extend g

By Transversality htpy thm:

$\exists$ $F: W \to Y$ htopic to G

s.t. $F \pitchfork Z$

Observe: ∂F is htopic to $\partial G = g$

so $I_2(\partial F, Z) = I_2(f, Z)$

$$\begin{aligned} &\quad \| \\ &\# \partial \left(F^{-1}(Z) \right) = 0 \mod 2 \end{aligned}$$

1-mfd w. bdy

$\square$

Note Intersection numbers are a generalisation of degrees:

$$I_2(f, pt) = \deg_2 f.$$

Def Let $f: X \to Y$ $g: Z \to Y$ be smooth maps, $\dim X + \dim Z = \dim Y$.

We define

$$I_2(f, g) := I_2(f \times g, \Delta)$$

where: $f \times g: X \times Z \to Y$,

$\Delta \subset Y \times Y$ the diagonal

<u>Exercise</u> When $f(X)$, $g(Y)$ are transversally intersecting submfds,

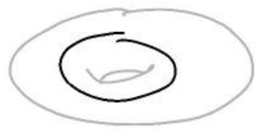
$$I_2(f,g) = \#\left(f(X) \cap g(Y)\right).$$

<u>Exercise</u> $I_2(f,g) = I_2(g,f)$.

<u>Thm</u> $I_2(f,g)$ is invt under htopies of f & g $\square$

(follows from above.)

<u>Example</u> Consider

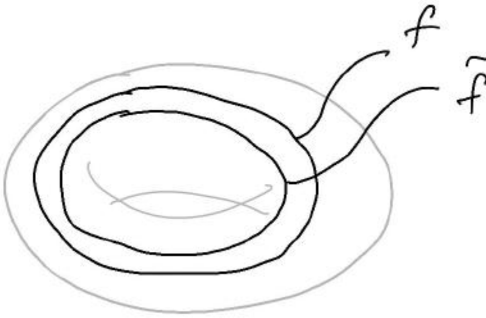
$f: S^1 \to T^2$ given by: 

$g: S^1 \to T^2$ given by: 

$I_2(f,f) = 0$

$I_2(g,g) = 0$

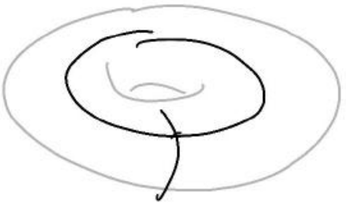
$I_2(f,g) = I_2(g,f) = 1$



$\tilde{f}$ homotopic to f

& $f(S^1) \cap \tilde{f}(S^1) = \emptyset$

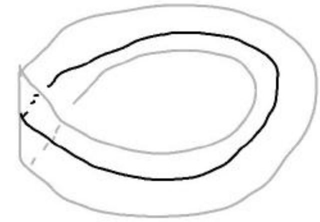
In particular, they are transverse so $I_2(f,f) = 0$



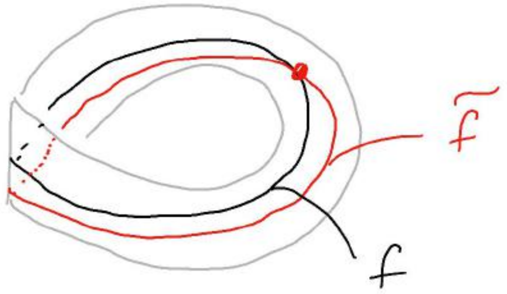
$f(S^1) \pitchfork g(S^1)$ at 1 pt

so $I_2(f,g) = 1$

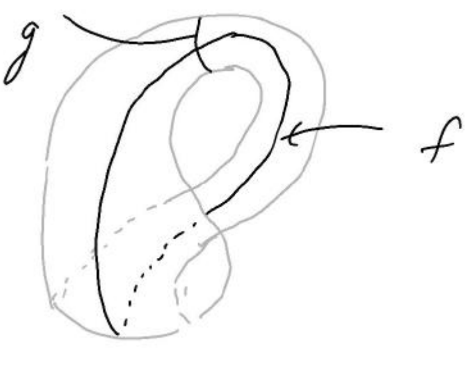
Example

$f: S^1 \to$ Möbius strip given by: 

Then $I(f,f) = 1$. Indeed, perturb f:



Example K = Klein Bottle; $f, g : S^1 \to K$:



$I(g,g) = 0$ because the nbhood of $g(S^1)$ is a cylinder;

$I(f,f) = 1$ because the nbhood of $f(S^1)$ is Möbius strip;

$I(f,g) = 1$ because $|f(S^1) \pitchfork g(S^1)| = 1$

$\boxed{\text{Winding numbers}}$

Suppose $\dim X = n-1$, $f: X \to \mathbb{R}^n$.
Assume $z \notin f(X)$ then define:

$$u : X \to S^{n-1}$$
$$u(x) = \frac{f(x) - z}{|f(x) - z|}$$

<u>Def</u> (wrapping number) $W_2(f, z) := \deg_2 u(x)$.

<u>Thm</u> Suppose $X = \partial W$ & $f = \partial F$
for some $F: W \to Y$ (<u>ie</u> F extends f).
Then:
$$W_2(f, z) = \# F^{-1}(z).$$

ie the number of "roots" $F = z$ in W is computed by the winding number.

Idea of proof

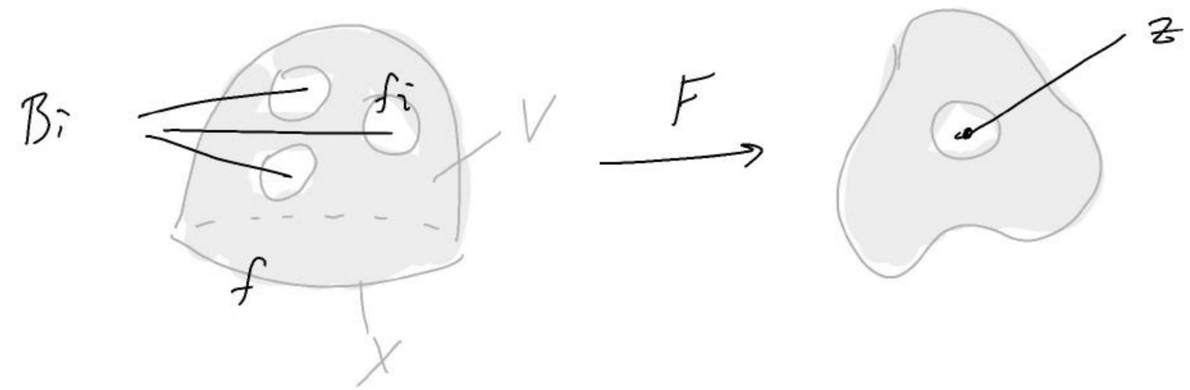
1) If $F \neq z$ on W, then $W_2(f, z) = 0$ because it extends to $\overline{W}$ by:

$$\frac{F-z}{|F-z|}$$

(see earlier thm)

2) Assume $F^{-1}(z) = \{y_1 \dots y_\ell\}$ are regular. Let B_i = open nbhood of y_i &

$$\overline{V} = \overline{W} \setminus (\sqcup B_i)$$



Then by 1), $W(f, z) = \sum W(f_i, z)$

where $f_i = F|_{\partial B_i}$

3) One can show $w(f_i, z) = 1$
because in a chart for B_i, $f_i = Id$
because it's a local diffeo.