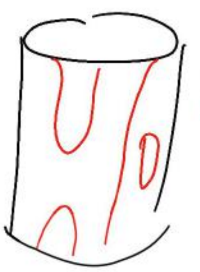
Lec 13 Recall last time:

- Defined what it means for $f, g : X \to Y$ to be homotopic: there must exist
$$F : X \times [0,1] \longrightarrow Y$$
st $F|_{X \times \{0\}} = f$, $F|_{X \times \{1\}} = g$.

- Defined mod 2 degree for $f : X \to Y$ if $\dim X = \dim Y$ & X is comp. without bdy:
$$\deg_2 f = \# f^{-1}(y) \mod 2$$
for any regular value $y \in Y$.

We proved that $\deg_2 f$ is homotopy invt.
Central idea:



$F^{-1}(y) =$ 1-dim'l mfd w. bdy — provided that y is also regular for F.

We will now prove a weak version of the Fundamental thm of algebra using degrees.

__Thm__ Suppose $\dim X = \dim Y$, $X = \partial W$, and $f : X \to Y$ is a smooth map.
If f __extends to__ W, meaning:

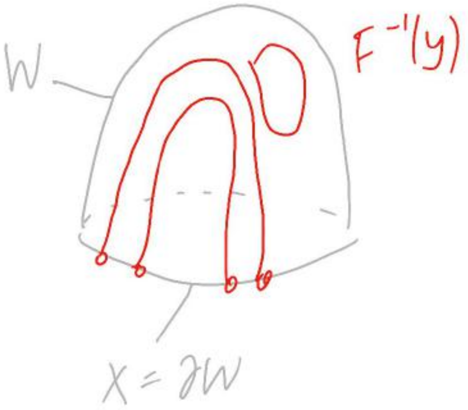
$$\exists\, F : W \to Y \quad \text{st} \quad F|_{\partial W} = f$$

then

$$\deg_{(2)} f = 0$$

will be true for $\mathbb{Z}$-degree, when we define it.

__Proof__ Pick $y \in Y$ regular for __both__ f and F,



then $F^{-1}(y)$ is a 1-dim comp mfd with bdy, and

$$\partial F^{-1}(y) = f^{-1}(y) \quad \text{- even \# of pts}$$

$\square$

Now suppose $P(z) : \mathbb{C} \to \mathbb{C}$ is a (smooth) fcn and $\mathcal{U} \subset \mathbb{R}^2$ is a domain, __ie__ 2-mfd with bdy

Suppose P has no zeroes on ∂U:

$$P(z) \neq 0 \quad \forall z \in \partial U$$

Then we can define:

$$\frac{P(z)}{|P(z)|} : \partial U \to S^1$$

Prop If $\deg_2 \frac{P(z)}{|P(z)|} \neq 0$ then P has a root in U

Proof If not, then $\frac{P(z)}{|P(z)|}$ extends to U, and get contradiction with thm above. $\square$

In fact after we define integral degree, one can show that

$$\deg \frac{P(z)}{|P(z)|} = \# \text{ of roots of } P \text{ in } U.$$

[provided P is holomorphic $\longrightarrow$ this will mean that all points $P^{-1}(y)$ have the same sign when degree is computed]

Suppose $P(z)$ is a poly:

$$P(z) = z^m + a_1 z^{m-1} + \ldots + a_m$$

<u>Claim</u> For $S^1_r = \{|z| = r\} \subset \mathbb{R}^2$ a circle with suff. large radius r,

$$\deg_2 \frac{p(z)}{|p(z)|} = m \mod 2$$

(Will be precisely m integrally).

<u>Proof</u> Take the htopy $p(z) \overset{p_t(z)}{\rightsquigarrow} z^m$

defined by:

$$p_t(z) = t\,p(z) + (1-t)\,z^m$$

("linear interpolation")

$$\ldots = z^m + t(a_1 z^{m-1} + \ldots + a_m) =$$

$$= z^m \underbrace{(1 + t(a_1/z + \ldots + a_m/z^m))}_{\substack{\text{close to } 1 \text{ and never } 0 \\ \text{when } |z| = r \text{ is large enough}}}$$

<u>So</u> the homotopy below is well-def:

$$\frac{p_t(z)}{|p_t(z)|} \quad \text{between} \quad \frac{p(z)}{|p(z)|} \quad \& \quad \frac{z^m}{|z|^m}$$

the map above clearly has degree m mod 2 over S^1_r because it's the same map as

$$\begin{array}{ccc} e^{i\varphi} & \longmapsto & e^{im\varphi} \\ S^1 & \longrightarrow & S^1 \end{array}$$

$\forall$pt is regular & has m preimages

Transversality & Nbhoods

<u>Transversality thm</u> Let $F: X \times S \to Y$ be a smooth map, only X is allowed to have bdy. Let's look at F as at a <u>family</u> of maps

$$f_s(x) = F(x,s) : X \longrightarrow Y \quad \text{where } s \in S \text{ fixed.}$$

Suppose $Z \subset Y$ is a submfd, F & ∂F are transverse to Z.

Then for almost every $s \in S$, f_s & ∂f_s are transverse to Z.

<u>Notation</u> $\partial(\text{map}) = \text{map}/_{\text{boundary}}$

<u>Recall</u> f_s & ∂f_s transverse to Z is the condition which guarantees $f_s^{-1}(Z) \subset X$ is a mfd with bdy.

<u>Proof</u> Denote $W = F^{-1}(Z) \subset X \times S$: this is a mfd w. bdy,

$$\partial W = W \cap \partial(X \times S).$$

Take the projection $\pi : X \times S \to S$ & consider restriction to W:

$$\pi : W \to S.$$

<u>Claim</u> $s \in S$ reg. value for $\pi : W \to S$ (resp $\partial\pi$)

$$\Updownarrow$$

f_s transv. to Z (resp. ∂f_s trans. to Z).

[when the claim is done, rest follows by Sard]

Proof of claim Assume $f_s(x) = z \in Z$.
Then $F(x,s) = z$ & by transversality:

$$dF_{(x,s)}\, T_{(x,s)}(X \times S) + T_z Z = T_z Y$$

So $\forall a \in T_z Y$ $\exists b = (w, e) \in$

$$\in T_{(x,s)}(X \times S) = T_x X + T_s S$$

st $dF(w,e) - a \in T_z Z$.

omit point (x,s) & other points

Want: $\forall a \in T_z Y$, to have $v \in T_x X$ –

st. $df(v) - a \in T_z Z$.

Observe $dF(w, 0) = df(w)$

So we need to prove that

$$\operatorname{Im} dF(TX \times TS) = \operatorname{Im} dF(TX \times \{0\}) \mod TZ$$

$$\begin{tikzcd}
 & TS & \\
TX \times TS \arrow[ur, "\pi"] \arrow[d, "dF"'] \arrow[r, phantom, "\supset"] & TW \arrow[u, "\pi"'] \arrow[d, "dF"] \\
TY \arrow[r, phantom, "\supset"] & TZ
\end{tikzcd}$$

regularity of s for $\pi: W \to S$

Indeed look at the diagram;
take $(w, e) \in TX \times TS$ any,

find $(u, e) \in TW \subset TX \times TS$ for some
other u & same e lying in TW
(exists by surjectivity of the upper arrows).

Now compute:

$$dF(w,e) = \underset{\underset{TZ}{\cap}}{dF(u,e)} + dF(w-u,0)$$

so $\operatorname{Im} dF(TX \times TS) = \operatorname{Im} dF(TX \times 0)$.

Now for "want" on prev. page, take $v = w-u$
and the claim ($\Rightarrow$ thm) is proved $\square$

Transversality thm $\rightsquigarrow$ "transverse maps are generic".

Def (ε-nbhood) let $Y \subset \mathbb{R}^N$ be comp, without bdy.
The set
$$Y^\varepsilon = \{\text{pts in } \mathbb{R}^N \text{ at dist} < \varepsilon \text{ from } Y\}$$
is called the ε-nbhood of Y

ε-nbhoods:

$S^1 \subset \mathbb{R}^2$ $\quad$ $S^1 \subset \mathbb{R}^3$

$\underline{\varepsilon\text{-nbhood thm}}$ For small enough ε, Y^ε is a mfd. For any pt $w \in Y^\varepsilon$, $\exists$ unique closest point in Y, denoted by $\pi(w)$.
This defines a map

$$\pi: Y^\varepsilon \to Y$$

which is a submersion.

<u>Note</u> The fibres (= preimages of pts) of π are diffeo^c to open disks of dim $= N - \dim Y$.
Some fibres are shown in gray above.

We will sketch a proof, and need some definitions first.

<u>Note</u> Y^ε is obviously a mfd — it's open (full dim.)

<u>Def</u> For $y \in Y \subset \mathbb{R}^N$, the normal space $N_y Y$ is the orthog complement of $T_y Y$ in $\mathbb{R}^N$:

$$N_y Y = (T_y Y)^\perp$$

<u>Note</u> This depends on the embedding $Y \subset \mathbb{R}^N$, even up to diffeomorphism!

<u>Def</u> The normal bdle $NY \subset \mathbb{R}^N \times \mathbb{R}^N$ is the set of points

$$\{(y, v) \in Y \times \mathbb{R}^N : v \in N_y Y\}$$

<u>Note</u> This is analogous to the def'n of TY from $T_y Y$.

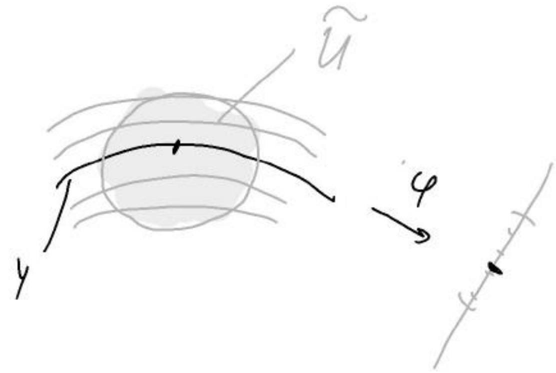
<u>Note</u> There is a projection $\sigma: NY \to Y$.

<u>Lemma</u> NY is a mfd, and $\sigma: NY \to Y$ is a submersion

<u>Proof</u> Define Y locally by equations: fix $y \in Y$ & find

$$y \in \underset{\text{open}}{\tilde{U} \subset} \mathbb{R}^N, \quad \underset{\text{submersion}}{\varphi: \tilde{U} \longrightarrow \mathbb{R}^k}$$

st $\varphi^{-1}(0) = \mathcal{U} \subset Y$ and is a chart for Y.



Then: $d\varphi_y : \mathbb{R}^N \twoheadrightarrow \mathbb{R}^K$ surjective

& $\operatorname{Ker} d\varphi_y = T_y Y$

Apply transpose, then locally:

$$(d\varphi_y)^t : \mathbb{R}^K \hookrightarrow \mathbb{R}^N \quad \text{injective}$$

$$\& \ \operatorname{Im} (d\varphi_y)^t = (T_y Y)^{\perp} = N_y Y.$$

(This is an ex. in lin. alg.)

<u>So</u> the map

$$\begin{array}{ccc} (y, v) & \longmapsto & (y, (d\varphi_y)^t(v)) \\ \cap & & \cap \\ \mathcal{U} \times \mathbb{R}^K & & N\mathcal{U} \subset NY. \end{array}$$

is a bijection & diffeo onto image

Therefore NU is a chart for NY,
and NY is a mfd

Finally, in this chart, 6 becomes a std projection
$$U \times \mathbb{R}^k \longrightarrow U$$
So 6 is a submersion $\square$

<u>Proof of ε-nbhd thm</u> (sketch)

Denote $h : N(Y) \longrightarrow \mathbb{R}^N$
$$h(y, v) = y + v.$$

(Translation of Y along the dir of v)

<u>Claim</u> h is a local diffeo (= submersion, since dims are same) at every pt of $Y \times \{0\}$.

<u>Proof</u> $T_{(y,0)}(NY) = T_y Y \oplus T_y Y^\perp = \mathbb{R}^N$

& dh becomes Id in this splitting $\square$

<u>So</u> h takes
some open nbhood of $(Y \times \{0\})$ in NY
<u>diffeomorphically</u> <u>onto</u>

Some open nbhood of Y in $\mathbb{R}^N$

[This follows from the fact h is a local diffeo — an exercise.]

<u>But</u> Any op. nbh. of Y in $\mathbb{R}^N$

$$\cup$$

$$Y^{\varepsilon} \quad \text{for suff. small } \varepsilon$$

<u>So</u> can take $h^{-1}(Y^{\varepsilon}) \subset NY$

& get the submersion $Y^{\varepsilon} \xrightarrow{\pi} Y$ by

$$\begin{tikzcd} Y^{\varepsilon} \arrow[r, "h^{-1}", "\cong"'] \arrow[dr, dashed, "\pi"'] & h^{-1}(Y^{\varepsilon}) \arrow[d, "\sigma"] \\ & Y \end{tikzcd} \quad \text{restricted from} \quad \begin{tikzcd} NY \arrow[d] \\ Y. \end{tikzcd}$$

We won't prove π has the explicit form in terms of closest points. $\square$

<u>Corollary</u> Let $f: X \to Y \subset \mathbb{R}^N$ smooth, Y without bdy. Then $\exists$ open ball $S \subset \mathbb{R}^N$ & a map $F: X \times S \to Y$