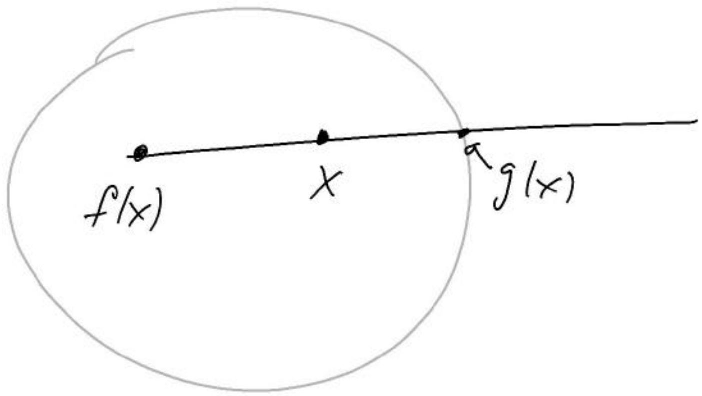
<u>Brouwer fixed pt thm</u> Any smooth map
$f: B^n \longrightarrow B^n$ has a <u>fixed pt</u>

unit ball $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$

<u>Proof</u> Suppose f has no fixed pts; then define

$$g: B^n \longrightarrow \partial B^n \text{ by :}$$



[Draw a ray from $f(x)$ to x & intersect with ∂B^n. The intersection pt is $g(x)$].

Exercise: g is smooth.
Clearly : g is a retraction: $g|_{\partial B^n} = Id$.

So we get a contradiction. $\square$

Homotopy and isotopy

Topology studies properties of maps which do not change under smooth /continuous deformations : seek to define invariants of smooth maps.

Let us make the notion of deformation precise

Def Two maps $f, g: X \to Y$ are called (smoothly) homotopic if there exists a smooth map

$$F: X \times [0,1] \to Y$$

such that $F(x,0) = f(x)$, $F(x,1) = g(x)$ $\forall x \in X$.

Notation: $f \sim g$.
The map F is called a homotopy from f to g.

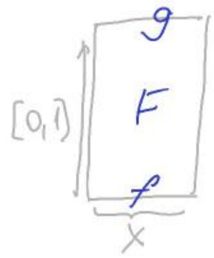
Equivalently a homotopy is a collection of maps

$$f_t : X \to Y \quad \text{for each } t \in [0,1]$$

such that $f_0 \equiv f$, $f_1 \equiv g$, and f_t <u>depend smoothly</u> on t.

The relation between the two definitions is that:

$$f_t \equiv F|_{X \times \{t\}} : X \to Y.$$



<u>Def</u> A map $f: X \to Y$ is <u>null-homotopic</u> if it is homotopic to a map $g: X \to Y$ which sends the whole of X to one point $\in Y$ (such g is called a <u>constant</u> map.)

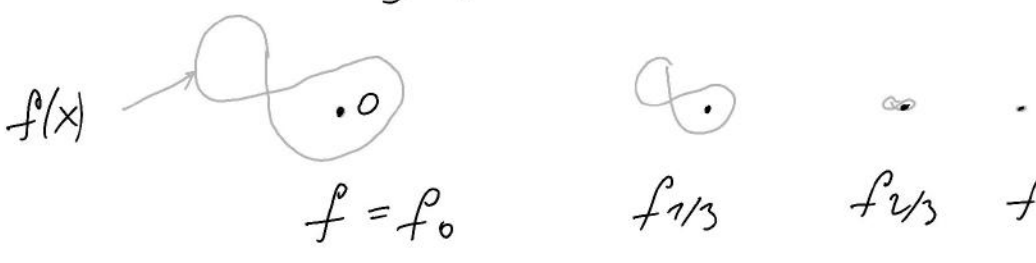
<u>Examples</u>

① Any map $f: X \to \mathbb{R}^n$ is null-homotopic.

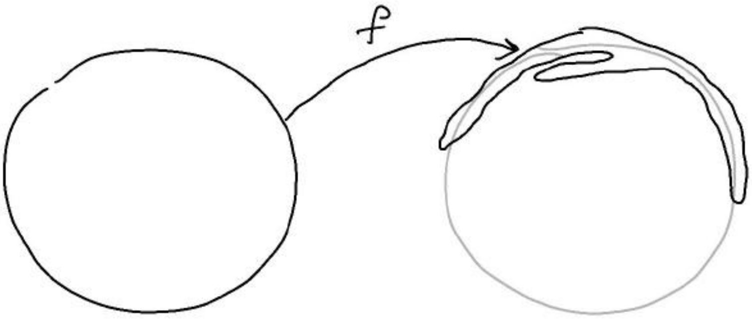
<u>Proof</u> Consider $f_t(x) = (1-t) \cdot f(x) \qquad t \in [0,1]$

multiplication of vector $f(x) \in \mathbb{R}^n$ by the scalar $1-t$

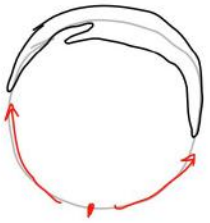
We see that $f_1 \equiv 0$. This homotopy is the <u>radial scaling</u>, picture:



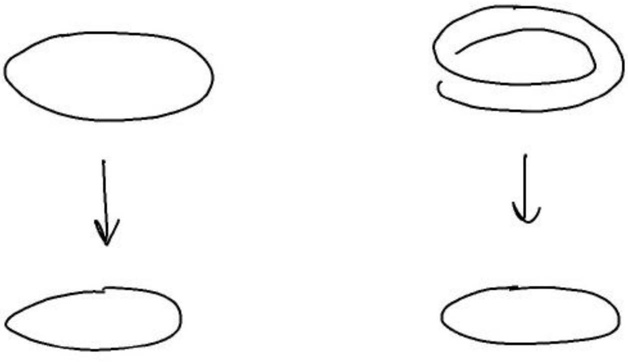
② The identity map $S^1 \to S^1$ is not null-homotopic (soon will show), but here's a map $f: S^1 \to S^1$ which <u>is</u> null-homot:



Actually, if $f: S^1 \to S^1$ is <u>not</u> surjective, then it is null-homotopic. Idea:

scale  from the missed point

③ Also:



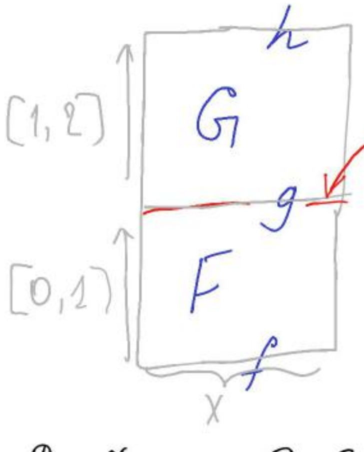
two maps $S^1 \to S^1$: they are not homotopic to each other, and not null-homotopic

Lemma Let $f, g, h : X \to Y$,
and $f \sim g$, $g \sim h$.
Then $f \sim h$.
(ie homotopy is an equivalence relation.)

Proof Denote the homotopies:

$$F : f \sim g$$
$$G : g \sim h$$

Both F, G are maps $X \times [0,1] \longrightarrow Y$.
Idea: stick them together.



This is $X \times \{1\}$. Here, $F(x,t)$ & $G(x,t-1)$ agree, and are equal to g.

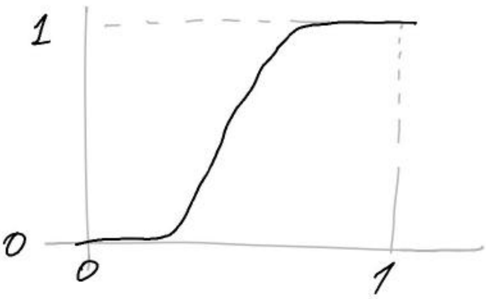
Problem: G & F may not glue smoothly on $X \times \{1\}$.
Solution: make $G(x,t-1)$, $F(x,t)$ t-independent near $t=1$.

Lemma We can modify $F(x,t) : X \times [0,1] \to Y$ to $\tilde{F}(x,t) : X \times [0,1] \to Y$ such that:

- we still have $\tilde{F}|_{X \times \{0\}} = f$, $F|_{X \times \{1\}} = g$

- $\tilde{F}(x,t)$ is t-independent for $t \leq \varepsilon$ and $t \geq 1-\varepsilon$.

<u>Proof</u> Consider $\tilde{F}(x,t) = F(x, \varphi(t))$ where $\varphi(t)$ is the following "time reparametrisation":



Graph of φ

$$\left.\begin{array}{l} \varphi(t) : [0,1] \to [0,1] \\ \quad \varphi \equiv 0 \quad , \; t \leq \varepsilon \\ \quad \varphi \equiv 1 \, , \quad t \geq \varepsilon \\ \varphi \text{ smooth.} \end{array}\right\}$$

Clearly, φ of this form exists. $\square$

Modify $G \rightsquigarrow \tilde{G}$ analogously. Now define:

$H : X \times [0,2] \to Y$ by:

$$\begin{cases} H(x,t) = \tilde{F}(x,t) & t \leq 1 \\ H(x,t) = \tilde{G}(x,t) & t > 1 \end{cases}$$

Then H is smooth & is a

homotopy from f to h. This proves $\sim$ is an equiv. relation. $\square$

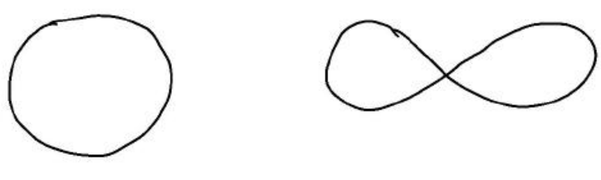
<u>Note</u> Formally, can reparametrise $H(x,t) \rightsquigarrow H(x,2t)$ so that in the end, $t \in [0,1]$ not $[0,2]$.

<u>Other notions</u>

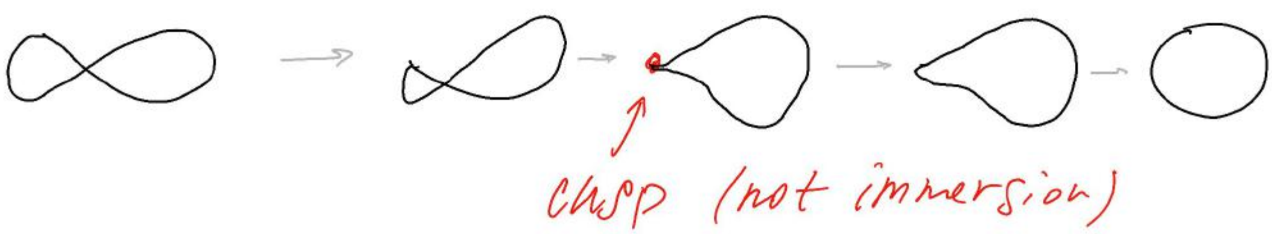
<u>Def</u>

If $f, g: X \to Y$ is a pair of embeddings / immersions then they are <u>isotopic</u> if there is a homotopy f_t between them such that each f_t is an embedding / immersion.

<u>Examples</u>

① 

Two immersions $S^1 \to \mathbb{R}^2$ <u>not</u> isotopic among immersions (but homotopic as smooth maps). Intuitively: a generic homotopy will require a <u>cusp</u>:



Degree of a map (mod 2 version)

Def Let $f: X \to Y$ be a map between manifolds of the same dim., X compact and without bdy. Pick a regular value $y \in Y$.

We define

$$\deg_2 f = \# f^{-1}(y) \mod 2$$

to be the number (mod 2) of preimage points of y (finite, as proved in assignments.)

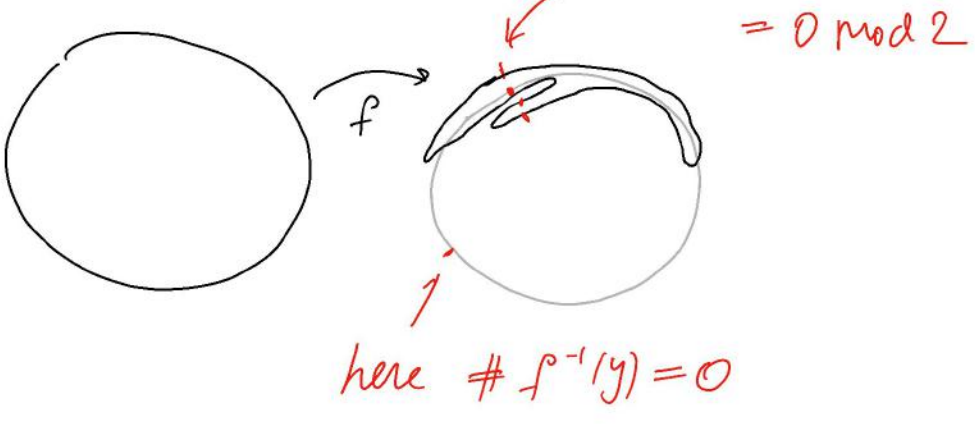
This is called the mod-2 degree of f,

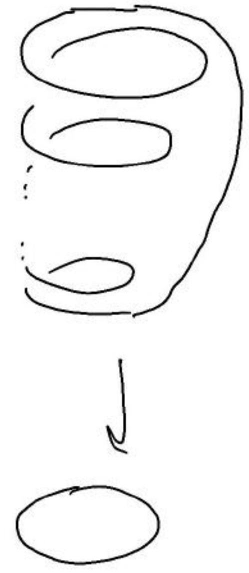
$$\deg_2 f \in \{0, 1\}.$$

Theorem (invariance) $\deg_2 f$ does not depend on:
① choice of regular value y

② homotopies of f.

Example





$k:1$ cover has degree k (mod 2)

We will prove the thm in several lemmas below.
Below, X & Y have same dim. (as in the thm.)

<u>Homotopy Lemma</u> Let $f, g : X \to Y$ be homotopic, and $y \in Y$ be regular value for f <u>and</u> g. Then

$$\# f^{-1}(y) = \# g^{-1}(y) \mod 2$$

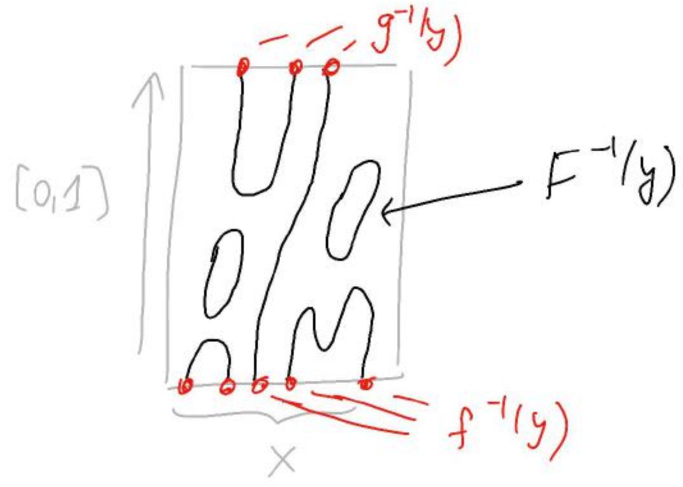
<u>Proof</u> Let $F : X \times [0,1] \to Y$ be a homotopy.
<u>Case 1</u> $y \in Y$ is also regular for F.
Then, by Preimage thm for mfds with bdy (prev. lec.),

$F^{-1}(y)$ is a comp. 1-diml mfd, and

$$\partial F^{-1}(y) = \left(f^{-1}(y) \times \{0\}\right) \sqcup \left(g^{-1}(y) \times \{1\}\right)$$

Therefore $\# f^{-1}(y) + \# g^{-1}(y)$ is even.

Picture:



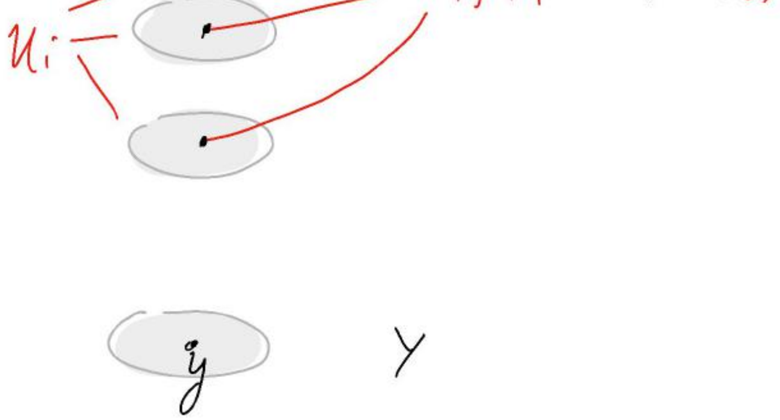
<u>Case 2</u> y not a reg value for F. Need several steps:

① "Stack of records thm": since y is regular for f, $\exists$ nbhood $U \subset Y$ of y such that

$f^{-1}(U)$ = disjoint union of nbhoods U_i of the points $p_i \in f^{-1}(y)$

and $f|_{U_i} : U_i \to U$ is a diffeo.

Picture:



The "stack of records" follows from <u>submersion thm</u> + <u>dim X = dim Y</u>, details are an exercise.

② Stack of records implies:
If y reg. value for f, then
$\forall y'$ close to y,

$$\# f^{-1}(y) = \# f^{-1}(y')$$ even integrally

③ Now, we are given y is reg. for f & g.
Find y' in a nbhood of y which is a <u>reg value</u> <u>for F</u> (exists by Sard), and use ② + <u>Case 1</u> □

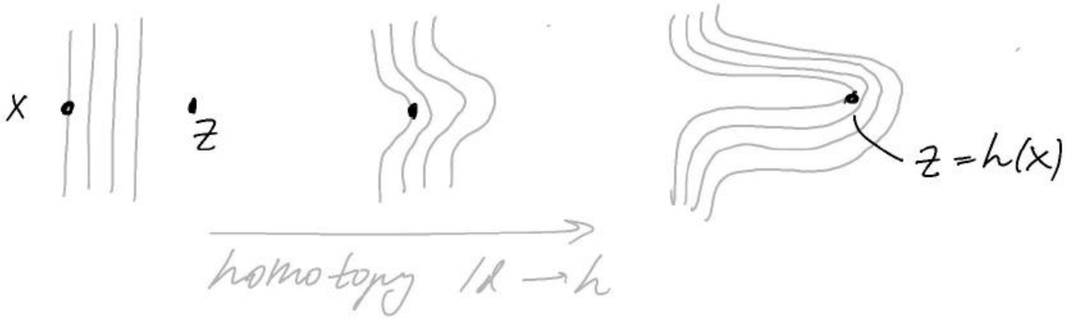
It remains to show $\deg_2 f$ does not depend on the choice of point y.

<u>Homogeneity lemma</u> Let $y, z \in Y$ be interior points in a smooth, connected mfd N.
Then there is a diffeo

$$h : N \to N$$ such that
$$h \sim Id$$ and
$$h(y) = z.$$ □

homotopic to Id

(This is intuitively obvious, we skip the proof.)



<u>Lemma</u> $\deg_2 f$ does not depend on the choice of reg value y.

<u>Proof</u> Suppose $y, z \in Y$ are two reg. values, take $h: Y \to Y$ from prev. lemma:

$$h(y) = z, \qquad h \sim Id.$$

Then:
$$\begin{aligned}
&\# f^{-1}(y) \\
&\quad \| \qquad\qquad\qquad \text{because } h(y) = z \\
&\# (h \circ f)^{-1}(z) \\
&\quad \| \qquad\qquad\qquad \text{because } h \circ f \sim f \\
&\# f^{-1}(z) \qquad\qquad \text{(since } h \sim Id\text{).}
\end{aligned}$$

$\square$

The invariance thm follows.

<u>Corollary</u> If X is a compact mfd without bdy, the <u>identity</u> map is <u>not</u> null-homotopic.

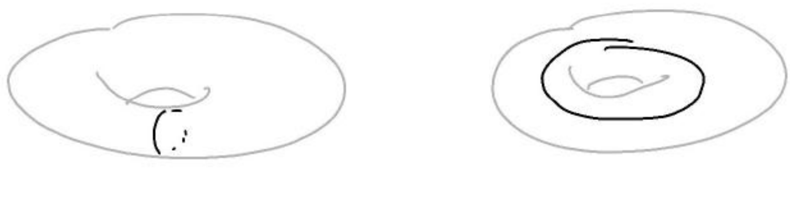
<u>Proof</u> $\deg_2 Id = 1$, $\deg_2 [\text{const. map}] = 0$ $\square$

<u>Note</u> If $f: X \to Y$ is not <u>onto</u>, then $\deg_2 f = 0$.

<u>Note</u> degree is an invariant for maps btw. mfds of the <u>same</u> dimension.

<u>Next</u>: intersection theory $\rightsquigarrow$ get invariants for maps $X \to Y$ where $\dim X \neq \dim Y$.

Prototypical example:



These two maps $S^1 \to T^2$ are <u>not</u> homotopic.