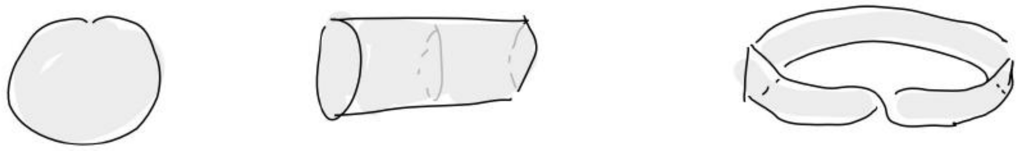
Manifolds with boundary

The disk, cylinder, Möbius strip have boundary and are not mfds in the sense of our earlier

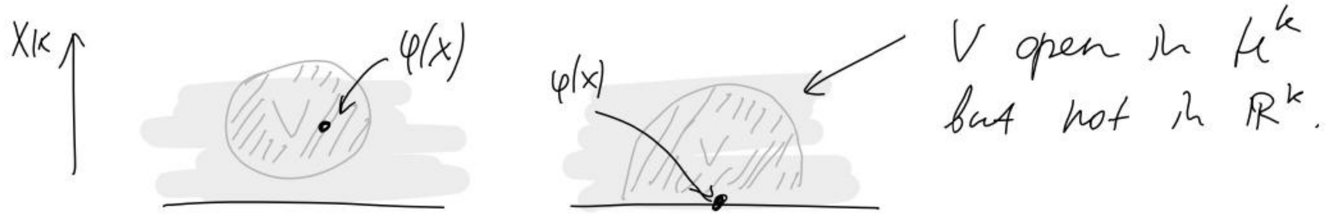


We need to give a separate def'n of a mfd with boundary

Notation: $H^k \subset \mathbb{R}^k$ is the upper half-space:
$H^k = \{(x_1 \dots x_k) \mid x_i \in \mathbb{R},\ x_k \geq 0\}$

Def A subset $X \subset \mathbb{R}^n$ is a k-dim'l mfd with bdy if $\forall$ point $x \in X$ $\exists$ nbhood U,
$x \in U \subset X$,
which is diffeomorphic to an open nbhood $V \subset H^k$.

Note: open nbhoods in H^k may look like:



Here "open" means in the topology in H^k, in particular V need not be open in $\mathbb{R}^k$

Def Take the diffeo $\varphi: U \to V$ from above. $x \in X$ is called an interior resp. boundary point if $\varphi(x)$ belongs to $\{x_k > 0\}$ resp. $\{x_k = 0\}$, see above fig.

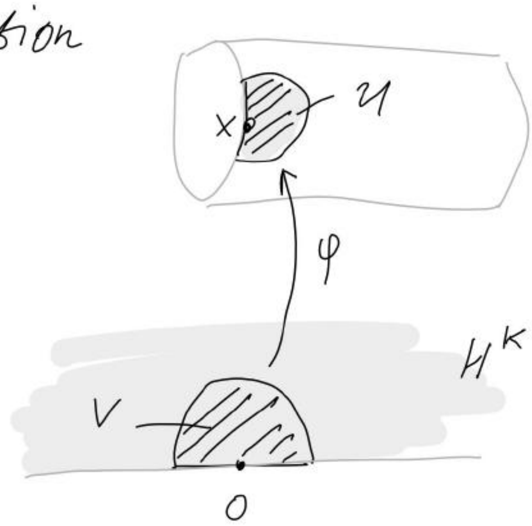
Def A compact manifold without boundary is called a closed mfd (to distinguish from a mfd w. bdy).

Claim If X, Y are mfds with bdy, $x \in X$, $f: X \to Y$ is a smooth map then:

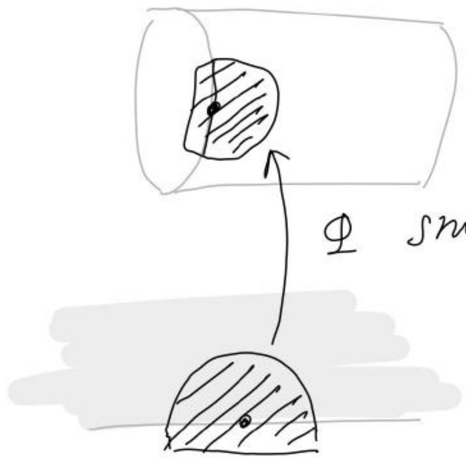
- $T_x X$ is well-defined
- df_x is well-defined.

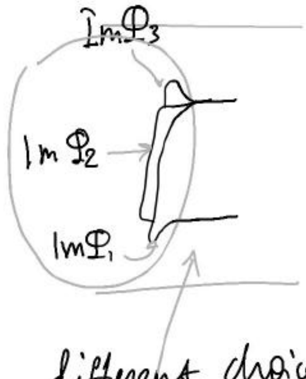
Sketch of proof If x is an interior point, this is the same as for mfds without bdy. When x is a boundary point, the definitions can be repeated with minor changes.

For example, to define $T_x X$, take a parametrisation



φ is a diffeo, in particular (by definition of a smooth map) it extends to a smooth map Φ defined on a nbhood of $0 \in H^k$:



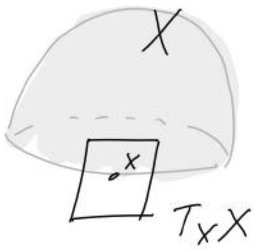


Now $d\Phi_0$ is well-defined (in terms of the matrix of partial derivatives) <u>so</u> can define $T_x X = \operatorname{Im} d\Phi_0$.

One has to prove as before that $T_x X$ does not depend on the choice of extension Φ;

we will skip these details. $\square$

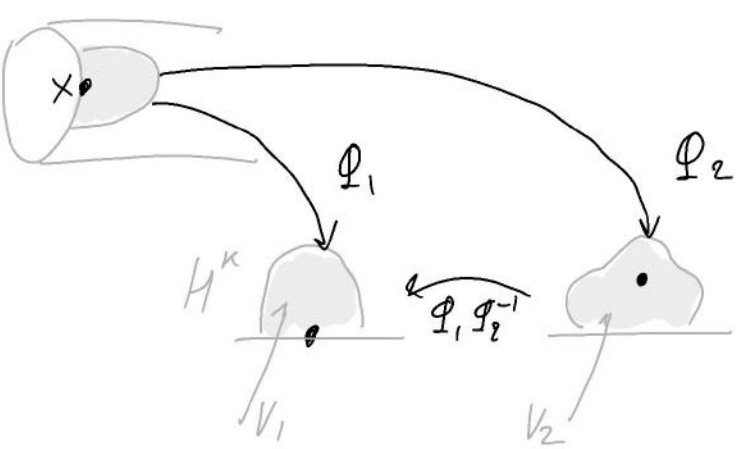
<u>Note</u> If $x \in X^k$ is a boundary pt, then T_xX is still a k-dimensional vector space



<u>Def</u> We denote by $\partial X \subset X$ the collection of all bdy pts of X. This is called the boundary of X.

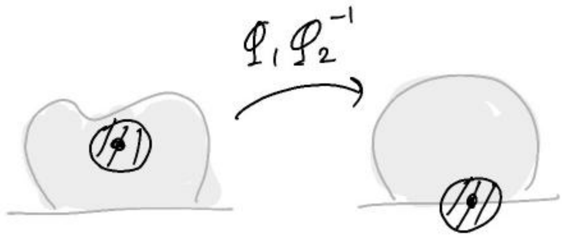
<u>Lemma</u> The notion of boundary point is well-defined, <u>ie</u> does not depend on the choice of φ above

<u>Proof</u> Suppose $x \in X$ & there exist different params φ_1, φ_2 s.t. $\varphi_1(x) \in \partial H^k$ and $\varphi_2(x) \in \operatorname{Int} H^k$



Consider $\varphi_1 \varphi_2^{-1}$: it is a diffeo $V_2 \to V_1$ taking $\varphi_2(x) \in \operatorname{Int} H^k$ to $\varphi_1(x) \in \partial H^k$.

Consider $\varphi_1 \varphi_2^{-1}$ as a submersion $V_2 \to \mathbb{R}^k$. It is an open map $\Rightarrow$ sends open sets to open sets $\Rightarrow$ its image contains a nbhood of $\varphi_1(x)$ in $\mathbb{R}^k$

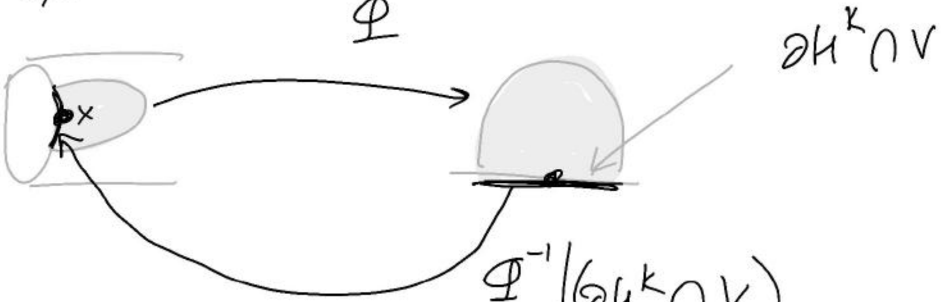


so $\operatorname{Im} \varphi_1 \not\subseteq H^k$, a contradiction $\square$

Corollary $\partial X \subset X$ is well-defined. $\square$

Corollary If X is a k-dim'l mfd w. bdy, ∂X is a $(k-1)$-dim'l mfd without bdy.

Proof $\varphi^{-1}|_{(\partial H^k \cap V)}$ provide a local parametrization for ∂X:



<u>Theorem</u> Let X be a mfd w. bdy, Y without bdy, f a smooth map, $Z \subset Y$ submfd without bdy.

Assume:

- f is transversal to Z
- $\partial f \underset{\text{def}}{=} f|_{\partial X}$ is transversal to Z.

Then $f^{-1}(Z)$ <u>is a manifold</u> with boundary, and

$$\partial(f^{-1}(Z)) = f^{-1}(Z) \cap \partial X,$$

& $\operatorname{codim}[f^{-1}(Z) \text{ in } X] = \operatorname{codim}[Z \text{ in } Y]$

Picture:

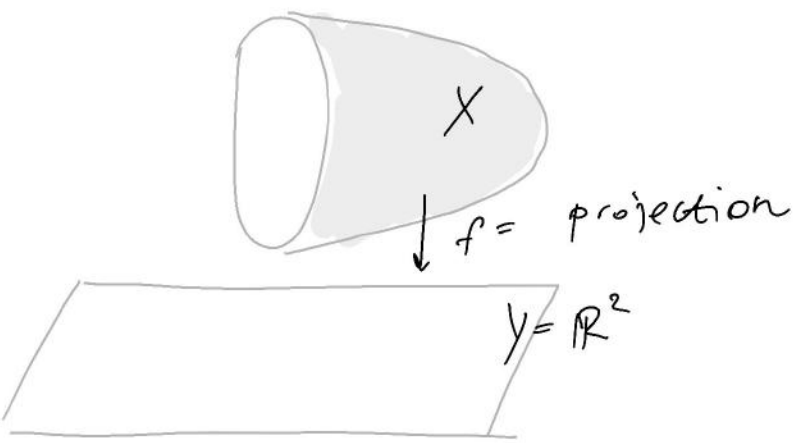
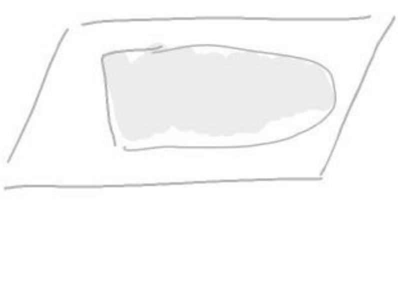
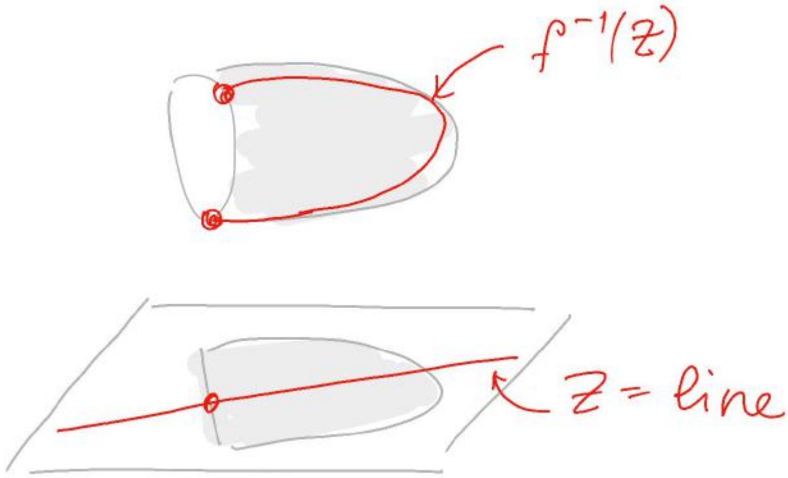
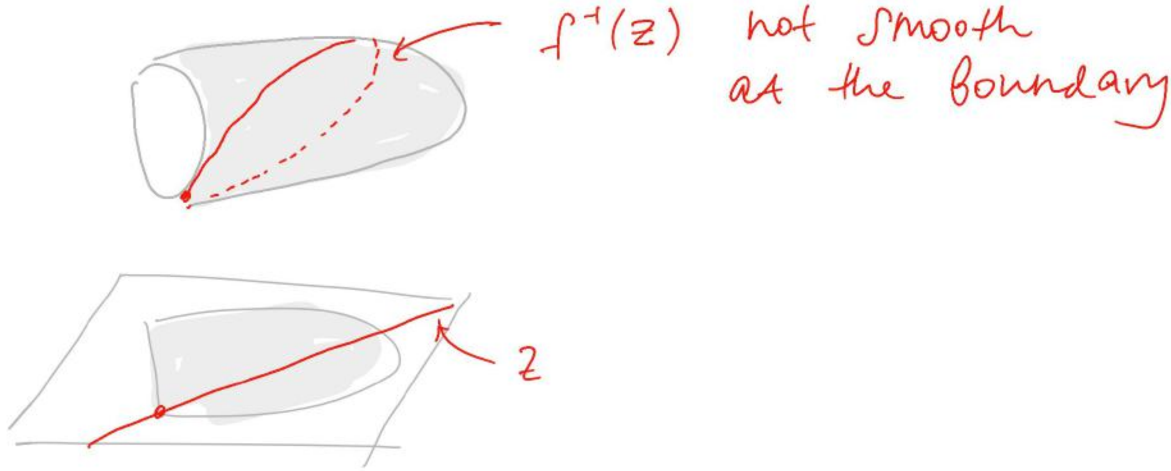


Image f:





Example when f transversal to Z
but $f|_{\partial X}$ is not:



Proof Int X $(= X \setminus \partial X)$ is a mfd without bdy $\Rightarrow$ $f^{-1}(Z) \cap \text{Int}\, X$ is a mfd (by submersion thm).

So: need to study $f^{-1}(Z)$ in a nbhood of $x \in \partial X \cap f^{-1}(Z)$

Claim It suffices to consider the case $Z = pt$.

Proof

such g exists as discussed previously

Indeed, pick a func $g : Y \xrightarrow{\text{submers.}} \mathbb{R}^{\dim Y - \dim Z}$ defined in a nbhood of $f(x) \in Z$ such that Z is locally given by:

$$Z = g^{-1}(0)$$

& consider $g \circ f$ instead of f.
The transversality conditions are equivalent to:

$$g \circ f, \quad g \circ f\big|_{\partial X} \quad \text{are } \underline{\text{submersions}} \text{ at } x$$

(check this) $\square$

So we are assuming $Z = pt$, and want to show $f^{-1}(pt)$ is a mfd w. bdy.

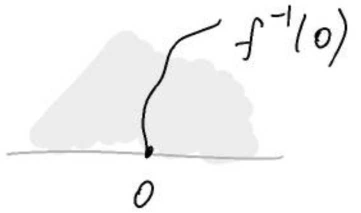
Next simplification: use local param φ & replace f by $f \circ \varphi^{-1}$

$$f \circ \varphi^{-1} : H^k \to \mathbb{R}^l$$
$$0 \mapsto 0 \quad \text{(can assume)}$$

[ $\longrightarrow \mathbb{R}^l$]

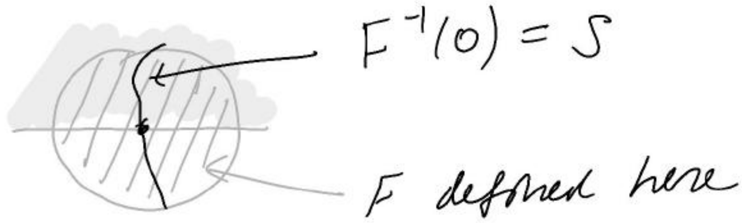
Now redenote $f \circ \varphi^{-1}$ by f again.

Want to show: $f^{-1}(0)$ is a mfd with boundary in $\mathbb{R}^k \supset H^k$



locally near 0

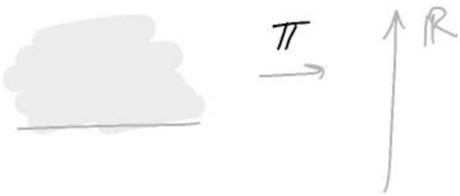
By def'n of smoothness, f locally extends to F defined in nbhood of $0 \in \mathbb{R}^k$:



- f is <u>submersion</u> at $0 \Rightarrow F$ is <u>submersion</u> at $0 \Rightarrow F^{-1}(0) = S$ is locally a mfd. without boundary

- We are also given that $f|_{\partial H^k}$ is a submersion. We will use this soon.

<u>Denote</u> $\pi : S \to \mathbb{R}$ be the projection to the last coord in H^k:

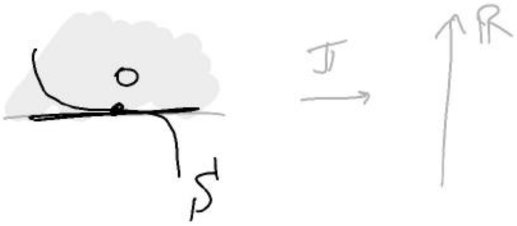


Then $f^{-1}(0) = \{ s \in S' : \pi(s) \geq 0 \}$

↳ recall: want to show this is a mfd w. bdy near 0.

<u>Claim</u> $0 \in S' \subset H^k$ is a reg value for π.

<u>Proof</u> Otherwise must have: $T_0 S'$ is horizontal, ie $T_0 S' \subset \mathbb{R}^{k-1} \times \{0\}$



Observe: $T_0 S' = \operatorname{Ker} dF_0 = \operatorname{Ker} df_0$
(follows generally from the fact that $S' = F^{-1}(0)$)
<u>So</u>
$\mathbb{R}^{k-1} \times \{0\} \subset \operatorname{Ker} df_0$.

<u>But</u> $\mathbb{R}^{k-1} \times \{0\} = T(\partial H^k) \Rightarrow f|_{\partial H^k}$ is <u>not</u>

a submersion at 0; this contradicts our hypothesis. □

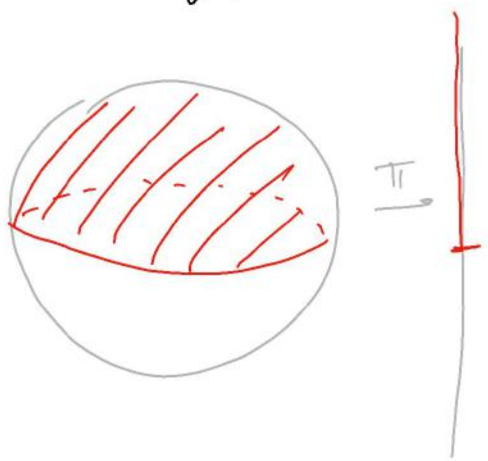
Finally, the thm follows from the lemma below; recall that $f^{-1}(0) = S \cap \{\pi \geq 0\}$. $\square$

<u>Lemma</u> Suppose S is a mfd without boundary & $\pi : S \to \mathbb{R}$ a smooth fcn with regular value 0. Then the subset

$$\{s \in S : \pi(s) \geq 0\}$$

is a mfd with bdy, and the bdy is $\pi^{-1}(0)$.

Picture:



<u>Proof</u> the set $\{\pi \geq 0\}$ is open $\Rightarrow$ is a mfd without bdy of the same dim. as S.
Suppose $s \in S$ & $\pi(s) = 0$. Near s, π can be brought (in some parametrisation) to the canonical projection form

$$(x_1 \dots x_k) \xrightarrow[\pi]{} x_k$$

so the set $\{\pi \geq 0\}$ is precisely $H^k =$
in this parametrisation $\square$

Sard's thm (for mfds w. bdy.)

X with bdy

Y without bdy

$f: X \to Y$ smooth $\Rightarrow$

almost every pt of Y is a reg value for both f and $f|_{\partial X}$.

Proof Because the union of two $\mu = 0$ sets has $\mu = 0$ $\square$

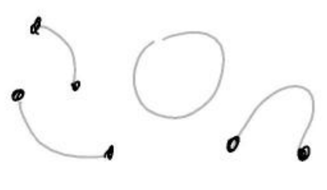
$\boxed{\text{One-manifolds}}$

This is intuitively obvious (rigorous proof omitted):

Thm $\forall$ compact connected 1-diml mfd is diffeomorphic to

$[0, 1]$ or S^1

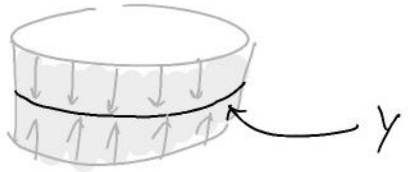
Corollary The boundary of a compact 1-diml mfd has an even number of pts.

Proof: obvious  $\square$

Fixed point theorems

<u>Def</u> If $Y \subset X$ are mfds, a <u>retraction</u> of X onto Y is a smooth map
$$g : X \to Y$$
s.t. $g|_Y = Id$.

<u>Eg</u> Cylinder retracts to circle:



<u>Theorem</u> If X is a <u>compact</u> mfd with bdy, there is no retraction $X \xrightarrow{g} \partial X$.

<u>Proof</u> Suppose g exists, take $z \in \partial X$ reg. value (exists by Sard). Then:
$g^{-1}(z) \subset X$ submfd w. bdy of dimension 1!, and

$$\partial\left(g^{-1}(z)\right) = g^{-1}(z) \cap \partial X = \{z\}$$

always true — because $g|_{\partial X} = Id$

so $\partial\left(g^{-1}(z)\right)$ is <u>one</u> point, which is impossible $\square$