Lec 09

<u>Thm</u> Any manifold has a Morse function.
More precisely, let $X \subset \mathbb{R}^n$ be a mfd, and $X_1 \dots X_n$ the std coords on $\mathbb{R}^n$.
Let $f : X \to \mathbb{R}$ be any function, and denote

$$f_a = f + a_1 X_1 + \dots + a_n X_n \quad : X \to \mathbb{R}$$

where $a = (a_i) \in \mathbb{R}^n$.
Then for almost every $a \in \mathbb{R}^n$,
f_a is a Morse fcn on X.

<u>Note</u> Almost every ... means: for every ..., <u>except</u> for a set of measure 0.

<u>Note</u> The above statement implies that $\forall f$ can be perturbed (by an arbitr. small amount) to become Morse.

<u>Lemma</u> Let $U \subset \mathbb{R}^k$ open, and $f : U \to \mathbb{R}$.
Then for almost all $a = a_i \in \mathbb{R}^k$,
$$f_a = f + a_1 X_1 + \dots + a_k X_k$$
is Morse on U

<u>Proof of Lemma</u> Consider

$$g : U \to \mathbb{R}^k$$
$$g = (\partial f/\partial x_1 \ldots \partial f/\partial x_k)$$

considered as a vector $(a_1 \ldots a_k)$

Now, $(d(f_a))_p$ equals:

$$\left(\frac{\partial f_a}{\partial x_1}\Big|_p, \ \ldots, \ \frac{\partial f_a}{\partial x_k}\Big|_p \right) = g(p) + a$$

<u>So</u>:

p is critical for $f_a \iff g(p) = -a$

<u>Also</u>:

$\mathrm{Hess}\,(f_a)_p = (dg)_p$ as $k \times k$ matrices

(because $f_a = f$ + linear terms).

<u>By Sard</u>:

<u>Almost every</u> a is a <u>regular value</u> for g.

<u>So</u> for the same points a,

$\mathrm{Hess}\,(f_a)$ never has $\det = 0$

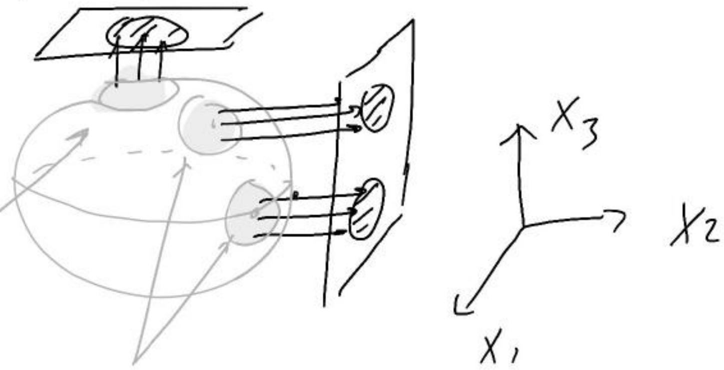
this means $\forall$ crit pt of f_a is non-degenerate $\square$

Lemma Let $X^n \subset \mathbb{R}^N$ be a n-dim'l mfd, $x \in X$. Then $\exists$ nbhood $x \in \mathcal{U} \subset X$ such & a subcollection $\{i_1 \dots i_n\} \subset \{1 \dots N\}$ such that the linear functions

$$x_{i_1} \;\dots\; x_{i_n}$$

are local coordinates on $\mathcal{U}$ (i.e. give a parametrisation of $\mathcal{U}$).

Picture:



$x_1 x_2$ are coords on this nbhood

x_1, x_3 are coordinates on these nbhoods

Proof $T_x X \subset \mathbb{R}^N$ is an n-dim'l subspace. Find indices $i_1 \dots i_n$ such that the projection $T_x X \longrightarrow \langle e_{i_1} \dots e_{i_n} \rangle$ is an iso

v. space gen. by basic vectors

(existence - by linear alg.)

Now, $x_{i_1} \dots x_{i_n}$ are coords on U

$\Updownarrow$

their restrictions of their differentials to $T_x X$ are linearly independent as maps $T_x X \longrightarrow \mathbb{R}$

ie together give a map $T_x X \to \mathbb{R}^n$ which is an isomorphism

But dx_i is precicely the projection onto the line $\mathbb{R} e_i$, so:

$$T_x X \xrightarrow[\text{proj}]{} \mathbb{R}\langle e_{i_1} \dots e_n \rangle \quad \text{iso}$$

$$\Updownarrow$$

$$x_{i_1} \dots x_{i_n} \text{ are local coords. } \square$$

Now we are ready to prove Thm above.

Proof Cover X by nbhoods U_α (finite or countable) st $\forall U_\alpha$ has local coords given by some std coord. fcns, as above.

Take one U_α, for concreteness assume that the first n coords in $\mathbb{R}^N$ are local coords for U_α:

$(x_1 \dots x_n)$ coord for U_α

Fix $c = (c_{n+1}, \dots, c_N)$ & define
$$f_{(0,c)} = f + \underbrace{c_{n+1} x_{n+1} + \dots + c_N x_N}_{\text{linear term}}$$

The first lemma implies:
for <u>almost all</u> $b = (b_1 \dots b_n) \in \mathbb{R}^n$,

$$f_{(b,c)} = f_{(0,c)} + b_1 x_1 + \dots + b_k x_k$$
is Morse on U_α.

Let $S_\alpha = \{ a \in \mathbb{R}^N$ s.t.
$f + a_1 x_1 + \dots + a_N x_N$ <u>not</u> Morse
on $U_\alpha \}$

We've just shown that $\forall c \in \mathbb{R}^{N-n}$,
$S \cap (\mathbb{R}^n \times \{c\})$ is <u>$\mu = 0$</u> in $\mathbb{R}^n$

So <u>by Fubini thm</u>:
S is $\mu = 0$.

So $S = \{ a \in \mathbb{R}^n$ s.t.
$f + a_1 x_1 + \dots + a_N x_N$ not Morse on
whole $X \} = \bigcup_\alpha S_\alpha$

is a countable union of $\mu=0$ sets,
so has $\mu=0$ $\square$

$\boxed{\text{Whitney embedding thm}}$

By defn, our manifolds always come embedded in some $\mathbb{R}^N$: $X^n \subset \mathbb{R}^N$.
Apriori, N can be very large.

Thm (Whitney embedding thm)
Every n-dimensional mfd admits an embedding into $\mathbb{R}^{2n+1}$.

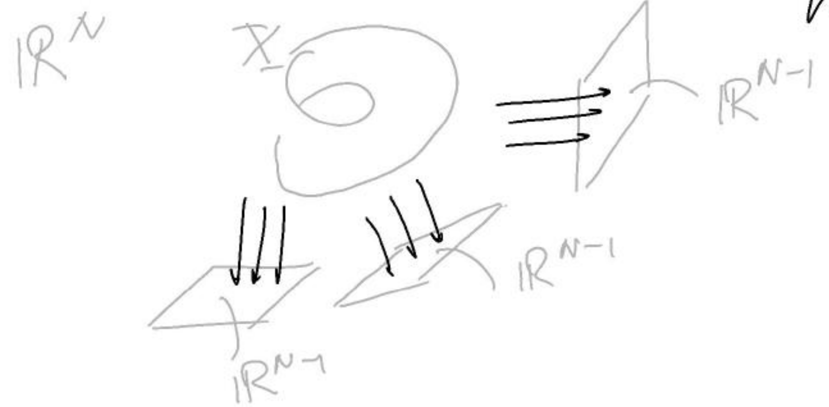
Why $2n+1$? This is the smallest dimension where 2 manifolds of dimension n generically don't intersect (or a single immersion generically doesn't self-intersect).

- Compare: $X^k, Y^m \subset \mathbb{R}^n$ submfds
  $\Rightarrow$ generically, $X \cap Y$ has dimension $\underline{k+m-n}$

Eg: $X^n, Y^n \subset \mathbb{R}^{2n}$: generic intersec is several points

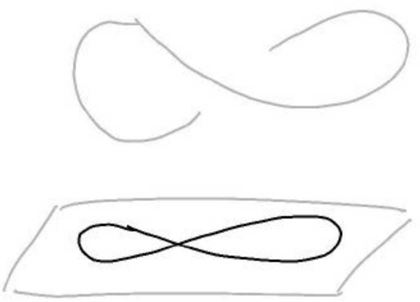
$X^n, Y^n \subset \mathbb{R}^{2n+1}$: generic intersec. is $\emptyset$.

- Idea of proof of Whitney thm: start with $X \subset \mathbb{R}^N$ & prove that for $N > 2n+1$, the projection to a <u>generic</u> <u>hyperplane</u> is an embedding when restricted to X



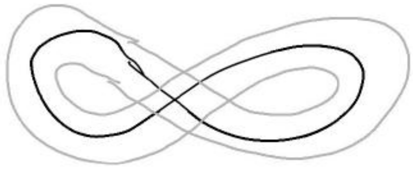
- Similarly can show: for $N > 2n$, the generic projection is an immersion (this is an exercise)

  <u>Eg</u> The generic projection of a <u>knot</u> $S^1 \subset \mathbb{R}^3$ is an immersed curve in $\mathbb{R}^2$;

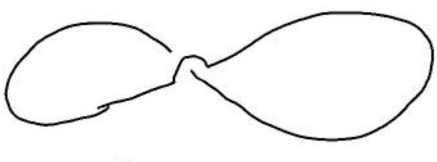


- The figure-8 in $\mathbb{R}^2$ cannot be <u>perturbed</u> to an embedding in $\mathbb{R}^2$

But can be perturbed to an embedding in $\mathbb{R}^3$.



perturbs. in $\mathbb{R}^2$ not embeddings



perturbation in $\mathbb{R}^3$ becomes an embedding

Because the proof of Whitney emb. thm uses "crude" genericity arguments, it frequently gives non-optimal bounds
(eg: S^1 embeds in $\mathbb{R}^2$ not only $\mathbb{R}^3$).

We move on to a proof.

Def Let $X^n \subset \mathbb{R}^N$ be a mfd.
Its tangent bundle is a (non-comp.) mfd
$$TX \subset \mathbb{R}^N \times \mathbb{R}^N$$
defined by:

$$TX = \{(x, v) \in X \times \mathbb{R}^N : v \in T_x X\}$$

This is the "union of all tangent planes put together"

<u>Def</u> If $f: X \to Y$ smooth map, its <u>differential</u> (or derivative) is a map

$$df : TX \longrightarrow TY$$

defined by:

$$df(\underset{T_xX}{\underset{\cap}{x, v}}) = (\underset{Y}{\underset{\cap}{f(x)}}, \underset{T_{f(x)}Y}{\underset{\cap}{df_x(v)}})$$

(Recall that df_x was defined much earlier).

<u>Lemma</u> df is a smooth map.

<u>Proof</u> Let $x \in U \subset \mathbb{R}^N$ be an open nbhood in $\mathbb{R}^N$ where f has a smooth extension

$$F : U \to \mathbb{R}^N$$

Then $$dF : \underset{U \times \mathbb{R}^N}{\underset{\parallel}{TU}} \longrightarrow \mathbb{R}^N \times \mathbb{R}^N$$

locally extends df & obviously is smooth, because the components of dF are
$\partial F_i / \partial x_j \quad i,j = 1 \ldots N$ which are smooth

Now, $\mathcal{U}\times\mathbb{R}^N$ is an open nbhood (in $\mathbb{R}^N\times\mathbb{R}^N$) for (x,v) for any v so dF is indeed an extension of df. $\square$

Quick excursion into Bundles

Note $df: TX \longrightarrow TY$ is actually a bundle morphism (or bundle map) which simply means df restricts to a linear map on the fibres:

$$T_x X \xrightarrow[\substack{df_x = \\ = (df)\big|_{\{x\}\times T_x X}}]{} T_y Y \quad \text{is linear.}$$

(We won't need this fact).

What is a bundle over a mfd X^n?
It's a manifold E with a submersion $E \xrightarrow[\pi]{} X$ s.t. the fibers of π are copies of $\mathbb{R}^k$. So locally $\forall x\in X$, $\exists$ nbhood $\mathcal{U}\subset X$

s.t. $\exists$ diffeo $\varphi_{\mathcal{U}}$ making the diagram commute:

$$\begin{tikzcd} \pi^{-1}(\mathcal{U}) \arrow[rr, "\varphi_{\mathcal{U}}", "\cong"'] \arrow[dr, "\pi"'] & & \mathcal{U}\times\mathbb{R}^k \arrow[dl, "\text{proj onto 1st factor}"] \\ & \mathcal{U} & \end{tikzcd}$$

Such ψ_U is called a local trivialisation of $E \to X$.
A bundle is called trivial if it has the form $E = X \times \mathbb{R}^k \xrightarrow[\text{proj}]{} X$.

Tangent bdles are usually not trivial (TS^2 isn't trivial).
We won't need these facts about bundles and continue by treating df as just a smooth map.

Obviously, we still have:

- Chain rule : $d(f \circ g) = df \circ dg$
- $d(f \circ f^{-1}) = df \circ d(f^{-1}) = Id$
  so if $f : X \to Y$ is a diffeo, then
  $df : TX \to TY$ is a diffeo
- If $x \in U \subset X^n$

$$\begin{array}{c} U \subset X^n \\ \varphi \downarrow \\ V \subset \mathbb{R}^n \end{array}$$

is a parametrisation, then

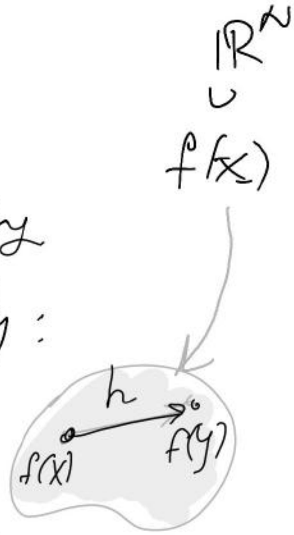
$$\begin{array}{c} TU = \{x \in U, v \in T_x X\} \\ d\varphi \downarrow \\ V \times \mathbb{R}^n \end{array}$$

is a parametrisation

∘ So TX^n is a smooth mfd of dim $2n$.

<u>Proof of Whitney thm</u> We'll prove that $\forall$ (not nec. compact) $X^n \subset \mathbb{R}^N$ admits an <u>injective immersion</u> in $\mathbb{R}^{2n+1}$.
For compact X, this is same as embedding; for non-compact, might get irrational winding etc. (Harder work required in non-comp. case, we'll skip it.)

Let $a \in \mathbb{R}^N$ be a unit vector, and $\Pi = a^\perp$ the orthog. hyperplane, $\Pi \cong \mathbb{R}^{N-1}$



Suppose $f : X \to \mathbb{R}^N$ is an embedding
Define $h : X \times X \times \mathbb{R} \to \mathbb{R}^N$ by:

$$h(x,y,t) = t\big(f(x) - f(y)\big)$$

"Gauss map"

and also consider $g : TX \to \mathbb{R}^N$

$$g(x,v) = df_x(v).$$

Assuming $\boxed{N > 2n+1}$ Sard's thm says:

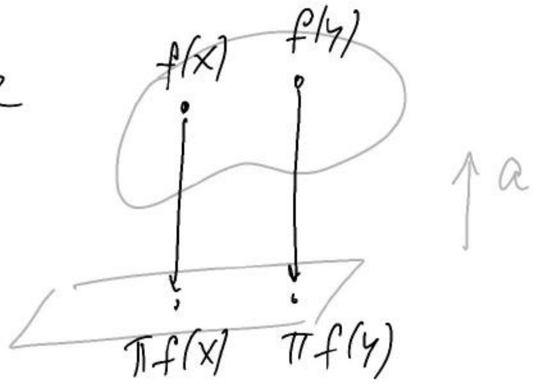
$\exists a \in \mathbb{R}^N$ such that
$a \notin$ Image h,
$a \notin$ Image g

Reminder: If $\dim X < \dim Y$
Sard's theorem says that
$\forall f: X \to Y$ smooth,
the image $f(X)$ has
measure 0.
(Because: reg. value $\equiv$ point
<u>not</u> in the image)

(Because h is a map from $2n+1$ dim'l mfd,
g —"— from $2n$ dim'l mfd)

Note: $a \neq 0$, because $0 \in$ Image h, Image g.
Now take $\Pi = a^\perp$
<u>Claim</u> The projection $\pi: \mathbb{R}^N \longrightarrow \Pi$
is an injective immersion on X.

<u>Proof</u>: ◦ Injectivity: Suppose
$\pi f(x) = \pi f(y)$



$\Updownarrow$

$f(x) = f(y) + ta$
for some $t \in \mathbb{R}$

$\Updownarrow$

$a \in$ Image (h)

◦ Submersion: suppose $d(\pi \circ f)_x$ has
nontriv kernel: $d(\pi \circ f)_x(v) = 0$.

By chain rule: $\pi \circ df_x(v) = 0$

π is already linear.

<u>So</u> $df_x(v) = ta$ for some $t \in \mathbb{R}$

$\Longleftrightarrow$ $a \in \text{Image}(g)$. $\square$