Lec 08 Last time:

- Proved Sard thm

- Defined the notion of $\underset{\sim}{\text{non-degen}}$ $\underset{\sim}{\text{crit pt}}$ $x \in X$ of a function $f: X \to \mathbb{R}$

Recall $x \in X$, $df_x = 0$, is called non-degen. if the Hessian matrix

$$\left( \frac{\partial^2 f}{\partial x_i \partial x_j} \Big|_x \right) \quad \text{has det} \neq 0$$

$\uparrow$ loc. coord. near $x$

Lemma Critical pts of a Morse func. are isolated

Proof Consider $g = (\partial f/\partial x_1 \ldots \partial f/\partial x_n)$

Then $dg_x = \text{Hess} f$ at $x$

$\Rightarrow$ $g$ is a local diffeo

$\Rightarrow$ $\underset{\text{Crit } f.}{\overset{\|}{g^{-1}(0)}}$ is one point (locally)

$\square$

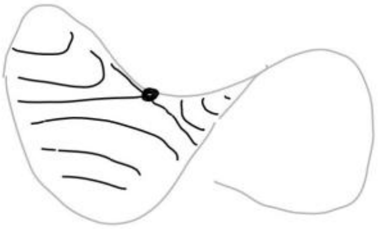
Morse lemma Suppose $x \in X$ is a non-degenerate crit. pt. of $f: X \to \mathbb{R}$, then there is a parametrisation of the nbhood of $x$

Such that in the local coordinates, $f$ is given by:

$$f = f(a) \pm x_1^2 \pm \dots \pm x_n^2.$$

We'll skip the proof.

The index of a non-degen. crit pt is the number of minuses in the canonical form (it's the same as the # of negative eigenvalues of Hess $f$ at $x$).



$f = x^2 + y^2$ $\qquad$ $f = x^2 - y^2$ $\qquad$ $f = -x^2 - y^2$

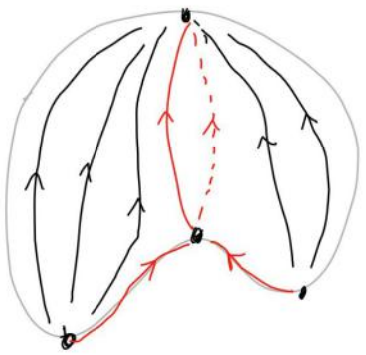
Morse theory says that any Morse function captures the topology of $X$.

Quick excursion: fix a Riem. metric $g$ on $X$ & a function $f : X \to \mathbb{R}$, then we have a canonical vector field on $X$, denoted $\nabla f$ & called the gradient.

Formally, it is given by:

$$\begin{tikzcd} T_xX^* \arrow[r, "g"] & T_xX \\ df_x \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \nabla f \end{tikzcd}$$

Geometrically, $\nabla f$ always points in the direction where $f$ increases most quickly:



There's always a <u>finite</u> # of flowlines between non-degen. crit. pts whose indices differ by 1

Counting these flowlines gives operations

$$\begin{tikzcd} \mathbb{R}[\text{Crit. pts of index } i] \arrow[r, "d"] & \mathbb{R}[\text{Crit pts of ind } i+1] \arrow[d, "d"] \\ & \cdots \end{tikzcd}$$

and $d^2 = 0$. This means that we can compute the homology of the above complex, and it gives the topological invariants $H^i(X)$.

Let's discuss some details of this story, we won't have proofs but will look at pictures. First, some algebra. Let $k$ be a field; we will assume char $k=2$ (eg $k=\mathbb{Z}/2$) so that we don't care about signs.

Def A chain complex is a collection of vector spaces $V^0, V^1, \dots V^n$ & linear maps $d_i : V^i \to V^{i+1}$ such that

$$d_{i+1} \circ d_i = 0.$$

This is sometimes abbreviated as "$d^2=0$" & we write $d$ for the whole collection of $d_i$'s,

eg: $dx$ means:
$d_i x$ if $x \in V^i$.

$d_{i+1} d_i = 0$ means: $\operatorname{Im} d_{i-1} \subset \operatorname{Ker} d_i$.

$$V^{i-1} \xrightarrow[d_{i-1}]{} V^i \xrightarrow[d_i]{} V^{i+1}$$

Def The homology of $(V, d) = (\{V^i\}, \{d_i\})$ is the collection of vector spaces $H^i(V)$ defined as:

$$H^i(V) = \frac{\operatorname{Ker} d_i}{\operatorname{Im} d_{i-1}} \longleftarrow \text{quotient of vector subspaces inside } V^i.$$

let $f: X^n \to \mathbb{R}$ be a Morse function.

Def The Morse (co)chain complex consists of the vector spaces $C^i(X, f)$

$$\begin{array}{c} C^i(X,f) \\ \| \\ \displaystyle\bigoplus_{\substack{p:\ \text{index } i \\ \text{crit pt of } f}} \mathbb{R}\langle p \rangle \end{array}$$

ie $C^i(X, f)$ is a vector space formally generated by the index $i$ crit pts of $f$.

To define the differential, we need the theorem below.

Thm The space of gradient flowlines between $p, q \in \mathrm{Crit}\, f$ is a mfd (possibly non-compact) of dimension

$$\operatorname{ind} q - \operatorname{ind} p - 1$$

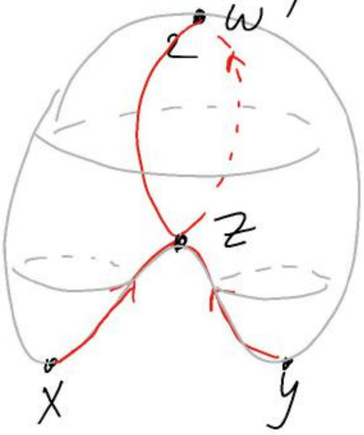
We denote this space by $\mathcal{M}(p, q)$.

Note More formally, $\mathcal{M}(p,q)$ consists of flowlines $u(t): \mathbb{R} \to X$ s.t.

$$u(t) \xrightarrow[t \to -\infty]{} p$$

$$u(t) \xrightarrow[t \to +\infty]{} q$$

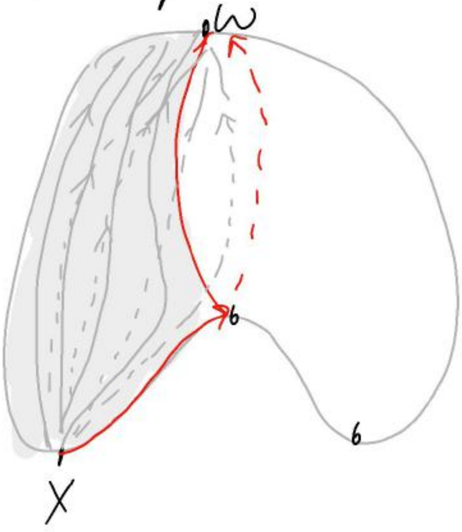
Here's a picture.



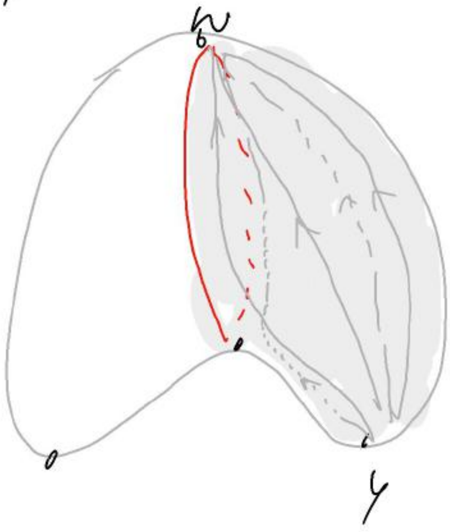
Finite # of flowlines

$x \to z$
$y \to z$
$z \to w$

Now, there are 1-dimensional spaces of flowlines $x \to w$, $y \to w$. They sweep 2-dim'l pieces of $S^2$:



$\mathcal{M}(x,w)$



$\mathcal{M}(y,w)$

The Morse differential $d(x)$ for $x \in C^i(X,f)$ counts flowlines from $x$ to all possible index $i+1$ crit pts:

$$d(x) = \left( \sum_{y \in C^{i+1}(X,f)} \# \mathcal{M}(x,y) \cdot y \right) \in C^{i+1}$$

this is the number of flowlines

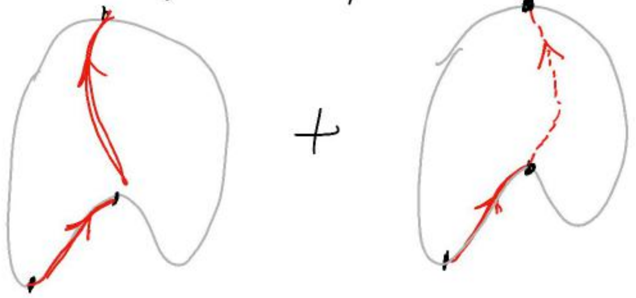
this expression is a linear combination of y's.

How do we prove $d^2=0$?

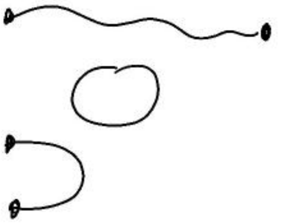
Thm let $x$: index $i$ crit pt,
$z$: ind. $i+2$ crit pt,
then $\mathcal{M}(x,z)$ is 1-dim'l & admits a compactification with boundary consisting of all pairs of flowlines of form:

$$x \longrightarrow \underset{\text{index } i+1}{y} \longrightarrow z$$

In the above picture $\partial\mathcal{M}(x,w) =$



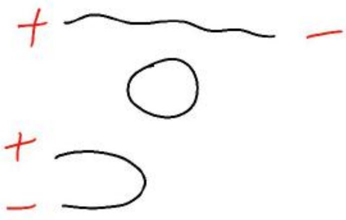
There are two such configurations, and in general $\partial\mathcal{M}(\underset{\text{index } i}{x}, \underset{\text{index } i+2}{z})$ has even number of points, because ∀ 1-dim'l compact mfd with boundary has even number of bdy points!



Now, $\underset{\text{index } i}{d^2(x)} = \sum_{\underset{\text{ind } i+1}{y},\ \underset{\text{ind } i+2}{z}} \#\mathcal{M}(x,y)\,\#\mathcal{M}(y,z)\cdot z$

$= \sum_{y,z} \#\partial\mathcal{M}(x,z)\cdot z = 0 \bmod 2.$

So $d^2=0$ over $\mathbb{Z}/2$. To prove over $\mathbb{Z}$ or non-char-2 field, must show that $\#\partial\mathcal{M}(x,z)=0$ when counted with signs



(It's quite hard to give rigorous definitions of signs).

Def The Morse homology $H^i(X,f)$ is the homology of the Morse complex $C^i(X,f)$ from above.

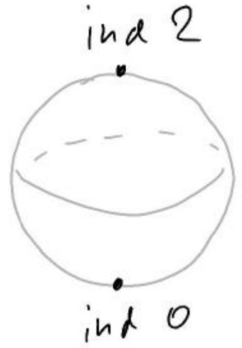
Thm The dimensions of the v. spaces

$$h^i(X) = \dim H^i(C,f)$$

do not depend on the choice of $f$, and are invariants of $X$ up to diffeomorphism

They are called Betti numbers.

Example $S^2$:



$C^2 = \mathbb{Z}/2$

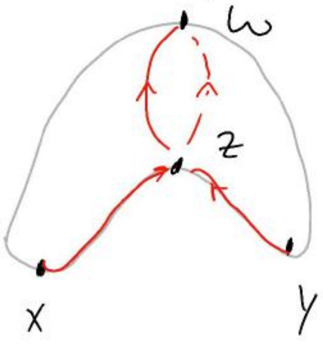
$C^1 = 0$

$C^0 = \mathbb{Z}/2$

can be no differential so $S^2$ has

| | |
|---|---|
| $h^2$ | 1 |
| $h^1$ | 0 |
| $h^0$ | 1 |

Another fction on $S^2$:



$w$

$z$

$X \quad Y$

$dz = w + w = 0.$

$dx = z$
$dy = z$

Compute homology:

$C^0$ : $d(x+y)=0$ & $x+y$ generate $\ker d_0$
so $H^0$ has dim $= 1$

$C^1$ : $dz=0$ But $z = dx$ so it is killed in $\operatorname{Ker} d / \operatorname{Im} d$ so $H^1 = 0$

$C^2$ : $dw=0$ & it is not in the image of $d$ so $H^2$ has dim $= 1$

We again have:

| | |
|---|---|
| $h^2$ | 1 |
| $h^1$ | 0 |
| $h^0$ | 1 |

Example Torus:

$dz = w + w = 0$

$dx = y + y = 0$

so $d \equiv 0$ & we have

| | |
|---|---|
| $h^2$ | 1 |
| $h^1$ | 2 |
| $h^0$ | 1 |

Computing flowlines may be hard in general & there are better tools for computing homology provided by algebraic topology. However, the Morse-theoretic approach to homology is extremely important; e.g it was used by Smale to prove the Poincaré conjecture in dim $\geq 6$

If $X^n$, $n \geq 6$, is simply-connected & has
$$H^0 = \mathbb{Z}$$
$$H^1 = 0$$
$$\vdots$$
$$H^{n-1} = 0$$
$$H^n = \mathbb{Z}$$

(must work over $\mathbb{Z}$)

then $X$ is homeomorphic to $S^n$.

This is not true for diffeomorphism: Kervaire & Milnor proved that $S^7$ has 28 smooth structures which are not diffeomorphic

We finish this informal discussion, and will now prove that Morse fns exist & are generic, using Sard

Thm Any manifold has a Morse function

More precisely, let $X \subset \mathbb{R}^n$ be a mfd, and $X_1 \dots X_n$ the std coords on $\mathbb{R}^n$.
Let $f: X \to \mathbb{R}$ be any function, and denote

$$f_a = f + a_1 X_1 + \dots + a_n X_n \quad : X \to \mathbb{R}$$

where $a = (a_i) \in \mathbb{R}^n$.
Then for almost every $a \in \mathbb{R}^n$,
$f_a$ is a Morse fcn on $X$.

Note Almost every... means: for every ..., except for a set of measure 0

Note The above statement implies that $\forall f$ can be perturbed (by an arbitr. small amount) to become Morse.

Lemma Let $U \subset \mathbb{R}^k$ open, and $f: U \to \mathbb{R}$.
Then for almost all $a = a_i \in \mathbb{R}^k$,
$$f_a = f + a_1 X_1 + \dots + a_k X_k$$
is Morse on $U$

Proof of Lemma  Consider

$$g : \mathcal{U} \to \mathbb{R}^k$$
$$g = (\partial f/\partial x_1 \ldots \partial f/\partial x_k)$$

Now, $(d(f_a))_p$ equals:

$$\left( \frac{\partial f_a}{\partial x_1}\Big|_p , \ldots , \frac{\partial f_a}{\partial x_k}\Big|_p \right) = g(p) + a$$

So:

$p$ is critical for $f_a \iff g(p) = -a$

Also:

$$\text{Hess}(f_a)_p = (dg)_p \quad \text{as } k \times k \text{ matrices}$$

(because $f_a = f +$ linear terms).

By Sard:

Almost every $a$ is a regular value for $g$.

So for the same points $a$,

Hess $(f_a)$ never has det $= 0$

this means $\forall$ crit pt of $f_a$ is non-degenerate $\square$