Lec 07

Recall from last lecture:

- Def'n of $f: X \to Y$ being transverse to $Z \subset Y$
- Def'n of what it means for $X, Z \subset Y$ intersect transversally: $X \pitchfork Y$
- Stated Sard's thm & discussed $\mu = 0$ sets. "measure zero"

Motivation for Sard: it allows to rigorously prove that various "transversality" properties as above are generic. We will have some examples later.

Sard's thm If $f: M^m \to N^n$ a smooth map, then the set of singular values $S \subset N$ has measure 0.

(Corollary: the set of reg. values $N \setminus S$ is dense in N)

See prev. lec. for discussion of measure 0

We discussed it's enough to prove a local version of Sard, by covering M by charts

Sard's thm, Local As above, but for $f: U \to \mathbb{R}^n$ where $U \subset \mathbb{R}^m$ is open.

Proof Denote C = crit. points, ie

$C = \{x \in U : df_x \text{ not onto}\}$,

then we want to show $S = f(C) \subset \mathbb{R}^n$ has $\mu = 0$

We define:

$$C \supset C_1 \supset C_2 \supset \dots$$

where:

$C_1 = \{x : df_x = 0\}$

$C_k = \{x :$ all partial derivatives of all components of $f = (f_1 \dots f_n)$ of order $\leq k$ vanish at $x\}$

Have 3 steps:

① $f(C \setminus C_1)$ has $\mu = 0$

② $f(C_i \setminus C_{i+1})$ has $\mu = 0$, $\forall i \geq 1$

③ $f(C_k)$ has $\mu = 0$, for large enough k.

<u>Proof of ①</u>

<u>Fubini thm</u>: if $A \subset \mathbb{R}^m$ is a set & $A \cap \mathbb{R}^{m-1} \times \{a\} \subset \mathbb{R}^{m-1}$ is $\mu = 0$ $\forall a \in \mathbb{R}$, then $A \subset \mathbb{R}^m$ is $\mu = 0$.

Now let $x \in C \setminus C_1$, so $\exists$ <u>some</u> nonzero derivative, say, $\partial f_1 / \partial x_1 \neq 0$ at x. (Notation: $f = (f_1 \dots f_n)$ as above).

It suffices to show: $\exists$ nbhood $V_x \ni x$, $V \subset \mathcal{U}$ s.t. $f(V \cap (C \setminus C_1))$ is $\mu = 0$

[Indeed, we can cover $\mathcal{U}$ by a countable set of of nbhoods V_{x_i}]

Consider $h(x) = (f_1(x_1 \dots x_m), x_2, \dots x_m)$
Then

$$dh_x = \left(\begin{array}{c|cccc} \neq 0 & ? & \dots & \dots & ? \\ \hline 0 & 1 & & & \\ \cdot & & 1 & & \\ \cdot & & & \ddots & \\ 0 & & & & 1 \end{array}\right) \quad \leftarrow \partial f_1 / \partial x_i$$

has full rank

$\Rightarrow h$ is a local diffeo $\Rightarrow$ has local smooth inverse.

Consider $g = f \circ h^{-1} : \underset{\text{some nbhood of } x}{V} \longrightarrow \mathbb{R}^n$, $V \subset \mathbb{R}^m$

S
$\|$

Claim $\mathrm{Sing.Values}(g) = \mathrm{Sing.Values}(f) \subset \mathbb{R}^n$

Proof: h local diffeo so by chain rule it takes:

$$\begin{tikzcd}
\mathrm{Crit\,Points}(f) \arrow[rr, "h"] \arrow[dr, "f"'] & & \mathrm{Crit\,Points}(g) \arrow[dl, "g"] \\
& S &
\end{tikzcd}$$

& diag commutes: $\square$

Returning to g:

Claim g has the form $(g_1 \ldots g_m)$ where $g_1(x_1 \ldots x_m) = x_1$.

Proof This agrees with $g \circ h = f$:

$$\underbrace{(x_1, \ldots)}_{g} \circ \underbrace{(f_1(x_1 \ldots x_m), \ldots)}_{h}$$

$$= \underbrace{(f_1(x_1 \ldots x_m), \ldots)}_{f}$$

$\square$

In full form,

$$g(x_1 \ldots x_m) = (x_1, g_2(x_1, x_2 \ldots x_m), \ldots, g_n(x_1, x_2 \ldots x_m)).$$

Note g preserves horizontal hyperplanes:

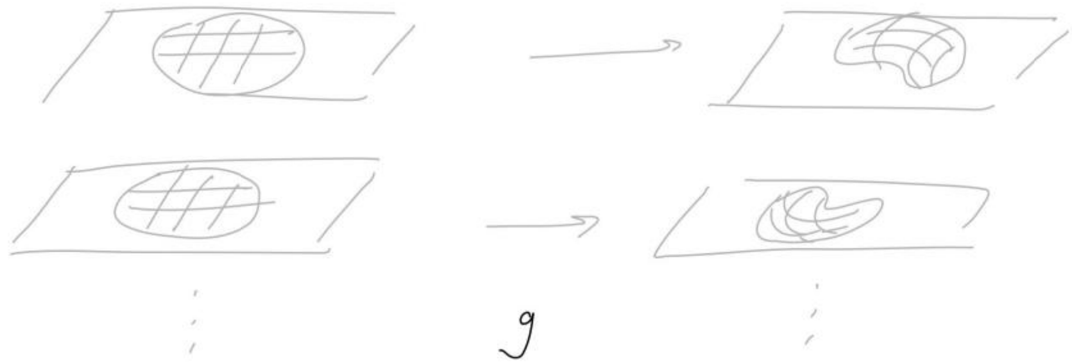
$$g(a, x_2 \dots x_m) = (a, g_2(a, x_2 \dots x_m), \dots)$$

consider as fixed param.

So can consider g as a family of smooth maps

$$\{a\} \times \mathbb{R}^{m-1} \xrightarrow{g^a} \{a\} \times \mathbb{R}^{n-1} \qquad a \in \mathbb{R}.$$

not a power, just notation



Claim B $(a, x_2 \dots x_m)$ is critical for g

$$\Updownarrow$$

$(x_2 \dots x_m)$ is critical for g^a.

Proof dg has the form:

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ ? & & & \\ \vdots & & \boxed{\;dg^a\;} & \\ ? & & & \end{pmatrix} \qquad \partial g_1/\partial x_i = \partial x_1/\partial x_i$$

It has full rank iff dg^a has full rank $\square$

So $\text{Sing. Values}(g) \subset \mathbb{R}^n$

$$\begin{array}{c} \text{Sing. Values}(g) \\ \| \\ \bigcup_{a \in \mathbb{R}} \text{Sing. Values}(g^a) \subset \mathbb{R}^{n-1} \times \{a\} \end{array}$$

By induction, these are $\mu = 0$

So $\text{Sing. Values}(g)$ is $\mu = 0$ by Fubini.
Step 1 is proved (see also Claim A.)

Proof of Step 2

let $x \in C_k \setminus C_{k+1}$, so some $(k+1)$'st derivative is nonzero while all k'th derivs. are zero

So there exists a k'th derivative

$$w = \frac{\partial^k f_r}{\partial x_{i_1} \ldots \partial x_{i_s}} : U \to \mathbb{R}$$

such that $w(x) = 0$ but $\frac{\partial w}{\partial x_{i_{s+1}}} \neq 0$.

Assume that $\frac{\partial w}{\partial x_1} \neq 0$. Similar to above, let

$$h(x_1 \ldots x_m) = (w(x_1 \ldots x_m), x_2, \ldots x_m) : U \to \mathbb{R}^m$$

then

h is a local diffeo since $dh_x = \begin{bmatrix} \neq 0 & ? & \ldots & ? \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix}$

<u>Claim</u> h carries $C_k \setminus C_{k+1}$ into the hyperplane $0 \times \mathbb{R}^{m-1}$.

<u>Proof</u> For $x \in C_k \setminus C_{k+1}$ we have: $w(x) = 0$ $\square$

Consider again $g = f \circ h^{-1} : V \to \mathbb{R}^n$
& denote $g^0 = g|_{\{0\} \times \mathbb{R}^{m-1}} : \mathbb{R}^{m-1} \to \mathbb{R}^n$

<u>By induction</u> Sing. Values (g^0) has $\mu = 0$ in $\mathbb{R}^n$.

But also:

$$\begin{array}{c} \text{Crit. pts}(g^0) \subset \{0\} \times \mathbb{R}^{m-1} \\ \cup\!\!| \\ h(C_k \setminus C_{k+1}) \end{array} \qquad (\bigstar)$$

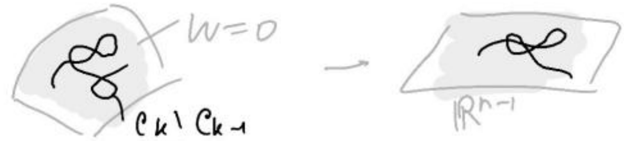
<u>Because</u> $x \in C_k$ is critical for f
$\Rightarrow h(x)$ is critical for g

Now take $(\bigstar)$ & apply g^0 to get:

$$\begin{array}{c} \mu = 0 \text{ set } \subset \mathbb{R}^n \\ \cup \\ g^0 h(C_k \setminus C_{k+1}) \\ \| \\ g h(C_k \setminus C_{k+1}) \\ \| \\ f(C_k \setminus C_{k+1}). \end{array}$$

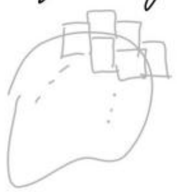
by Claim

Quick summary of Step 2: we proved that $C_k \setminus C_{k-1} \subseteq \{w = 0\}$ & the level set $\{w = 0\}$ is regular because $\partial w / \partial x_1 \neq 0$, so we can "flatten it up" by a diffeo.



<u>Note</u> Formally, we proved that $f((C_k \setminus C_{k+1}) \cap V)$ is $\mu = 0$ for some nbhood V of x, but then use covers.

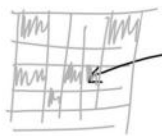
Proof of Step 3 Let $I \subset U$ be a cube with edge δ.
Can cover U by countable # of cubes $\Rightarrow$
suffices to prove $f(C_k \cap I)$ has $\mu = 0$



order $k+1$ term

Use Taylor: $f(x+h) = f(x) + R(x,h)$
where $\|R(x,h)\| \leq c\|h\|^{k+1}$
for $\boxed{x \in I \cap C_k}$, $x+h \in I$ (1)

Now subdivide I into r^n cubes w. edge δ/r



I_1 (Some cubes contain a pt in C_{k+1}, but maybe not all).

Assume $x \in I_1$; then any point in I_1 is given by:
$x+h$, $\|h\| \leq \sqrt{n}\ \delta/r$ (2)

By (1), $f(I_1) \subset$ Cube w. edge a/r^{k+1},
$a = 2c(\sqrt{n}\ \delta)^{k+1}$ is a const.

only true if I_1 contained $x \in C_k$.

So $f(C_k \cap I) \subseteq$ Union of $\leq r^n$ cubes
with total volume $V \leq$
$\leq r^n (a/r^{k+1})^p = a^p r^{n-(k+1)p}$

Here $a = \text{const}$; now take k such that

$$n - (k+1)p < 0$$

then $V \xrightarrow[r \to 0]{} 0$.

Because r in our argument is arbitrary, the proof is complete $\square$

We will have two applications of Sard's thm:

- Existence of Morse functions
- Whitney embedding theorem

$\boxed{\text{Morse functions}}$

Suppose $f: X \to \mathbb{R}$ is a function. If X is compact, f must have critical points (eg: global max & global min must be critical pts).

crit pts of height fctn.

Morse fctns is a natural class of "generic" functions whose singularities have simplest form.

Def let $x \in X$ be a crit pt of f, ie $df_x = 0$
We say x is a nondegenerate crit pt
if $\exists$ parametrisation

$(x_1 \dots x_n)$ of the nbhood of $x \leftrightarrow (0, \dots 0)$

in which the Hessian matrix

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_0 \right)_{i,j=1\dots n}$$

is nondegenerate (ie has max. rank $\Leftrightarrow \det \neq 0$).

Lemma The defn does not depend on the choice of parametrisation.

Proof $\forall$ other param $(x_1' \dots x_n')$ differs from $(x_1 \dots x_n)$ by a local diffeo $\mathbb{R}^n \to \mathbb{R}^n$ (defined in nbhood of 0) :

$$x_1' = x_1'(x_1 \dots x_n)$$
$$\vdots$$
$$x_n' = x_n'(x_1 \dots x_n).$$

chain rule, summation by k & l

Then $\dfrac{\partial^2 f}{\partial x_i \partial x_j} = \dfrac{\partial}{\partial x_i}\left(\dfrac{\partial f}{\partial x_j}\right) = \dfrac{\partial}{\partial x_l'} \dfrac{\partial x_l'}{\partial x_i} \left(\dfrac{\partial f}{\partial x_k'} \dfrac{\partial x_k'}{\partial x_j} \right) =$

$$= \underbrace{\frac{\partial^2 f}{\partial x'_l \partial x'_k}}_{\substack{\text{Hess } f \\ \text{in } x'\text{-coords}}} \underbrace{\frac{\partial x'_k}{\partial x_j} \frac{\partial x'_l}{\partial x_i}}_{d(x \mapsto x')} + \frac{\partial f}{\partial x'_k} \cdot (\cdots)$$

vanishes at 0 = crit pt of f.

It follows that Hess f computed in x_i-coords & Hess f computed in x'_i-coords are matrices different by multiplication with the Jacobian $(\partial x'_i / \partial x_j)_{i,j=1\ldots n}$

non degenerate, because it's the differential of a local diffeo $\square$