Transversality

Recall: $f: X \to Y$, $\dim X \geq \dim Y$ then

$f^{-1}(y) \subset X$ is a mfd if y is a regular value

Now: let $Z \subset Y$ be a submfd.

Question: when is $f^{-1}(Z) \subset X$ a submfd?

Note: we don't require $\dim X \geq \dim Y$ any more

Definition $f: X \xrightarrow[\text{smooth}]{} Y$ is said to be

transverse to a submfd $Z \subset Y$ if

$\forall x \in f^{-1}(Z)$, $y = f(x)$, we have

$$(\star) \qquad \text{Image}(df_x) + T_y(Z) = T_y(Y).$$

this is an equality of vector spaces, and it means that $\text{Img}(df_x)$ and $T_y Z$ span $T_y Y$

Note We do not require $\text{Image}(df_x) \oplus T_y Z = T_y Y$

ie $\text{Image}(df_x) \cap T_y Z$ may be non-zero

Example $Z = y$ a point, then $(\star)$ becomes:

df_x is onto,

the definition of submersion

Theorem If $f: X \to Y$ is transverse to a submfd $Z \subset Y$, then

$$f^{-1}(Z) \text{ is a submfd of } X,$$

and

$$\text{codim}\left(f^{-1}(Z) \subset X\right) = \text{codim}\left(Z \subset Y\right)$$

Proof Denote the dimensions: X^n, $Z^k \subset Y^l$.

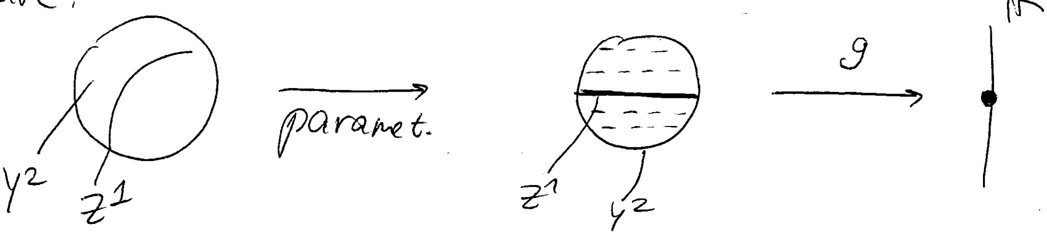
Find a parametrisation $(y_1 \dots y_l)$ near $y \in Y$ such that Z is a piece of $\mathbb{R}^k$ spanned by $(y_1, \dots y_k)$ ie given by $\{y_{k+1} = \dots = y_l = 0\}$.

Denote $g: \mathbb{R}^l \to \mathbb{R}^{(l-k)}$

$$g(y_1 \dots y_l) = (y_{k+1}, \dots, y_l) \qquad \text{(projection)}$$

then $Z = g^{-1}(0)$

Picture:



Now, $f^{-1}(Z) = (f \circ g)^{-1}(0)$

so must prove that 0 is a rg value for

$$f \circ g : X^n \longrightarrow \mathbb{R}^k$$

$$X^n \xrightarrow{f} Y^l \xrightarrow{g} \mathbb{R}^{l-k}$$

Consider the differentials:

$$\begin{tikzcd}
T_x X \arrow[r, "df_x"] & T_y Y \arrow[r, two heads, "dg_y"] & \mathbb{R}^{l-k} \\
& T_y Z \arrow[u, phantom, "\cup" description] \arrow[r] & 0
\end{tikzcd}$$

We know: dg_y is surjective (because g is a std projection)

Want: $dg_y \circ df_x$ is surjective.
Suppose

$$\text{Image}(df_x) + T_y Z = T_y Y,$$

using dg_y surjective

then

$$dg_y(\text{Image } df_x + T_y Z) = dg_y(T_y Y) = \mathbb{R}^{l-k}$$

$$= dg_y(\text{Image } df_x)$$

using that $dg_y|_{T_y Z} = 0$

So $\text{Image}(dg_y \circ df_x) = \mathbb{R}^{l-k}$

So $dg_y \circ df_x$ is surjective. $\square$

<u>Intersections of submanifolds</u>

Recall that: given $Z \subset Y$ submfd,
$f: X \to Y$,
we have a definition of f being transverse to Z.

Now assume that f is an <u>embedding</u>, so that $X \subset Y$ is <u>also</u> a submanifold.

Def Two submanifolds $X, Z \subset Y$ intersect transversally if

$$\forall x \in X \cap Z, \quad T_x X + T_x Z = T_y Y$$

Theorem If $X, Z \subset Y$ intersect transversally & $X \cap Z \neq \emptyset$, then $X \cap Z \subset Y$ is a submanifold, and

$$\operatorname{codim}_Y (X \cap Z) = \operatorname{codim}_Y X + \operatorname{codim}_Y Z.$$

Here, $\operatorname{codim}_Y *$ means $\dim Y - \dim *$.

Proof Consider the inclusion map $f: X \to Y$, then

$$T_x X = \operatorname{Image}(d f_x)$$

So f is transverse to Y, ~~to~~ and it follows from prev. theorem that

$$f^{-1}(Z) \subset Y \text{ is a submfd}$$

But $f^{-1}(Z) = X \cap Z$, obviously.
For the dimensions, check it yourselves using the prev. thm $\square$

Corollary If $X^n, Z^m \subset Y^k$ are submanifolds intersecting transversally and $x \in X \cap Z$ then there is a ~~chart~~ parametrization of Y near x with coords

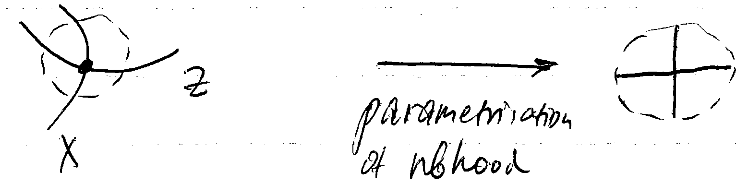
$$y_1 \ldots y_k$$

~~where~~ in which X and Z are pieces of linear subspaces given by:

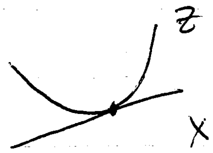
$$X = \{ y_{n+1} = \ldots = y_k = 0 \} \quad \text{(First } n \text{ coords. are free)}$$

$$Z = \{ y_1 = \ldots = y_{k-m} = 0 \} \quad \text{(Last } m \text{ coords. are free)}$$

Pictures • $\dim X = \dim Z = 1$, $Y = \mathbb{R}^2$:

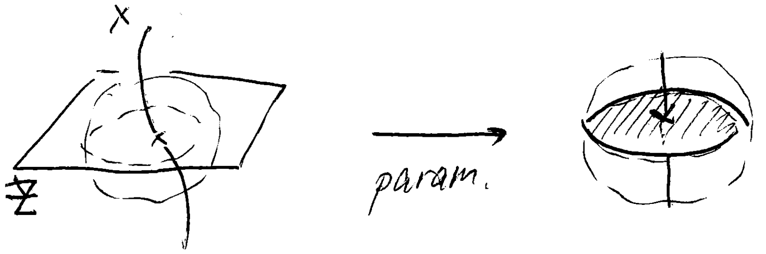


This is a transverse intersec

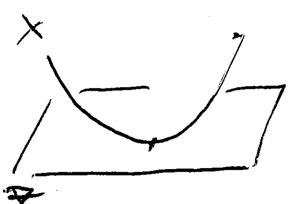


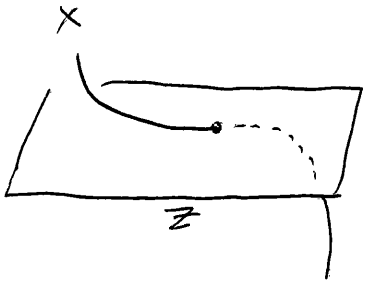
This is a non-transverse intersection.

- $\dim X = 1, \quad \dim Z = 2 \qquad Y = \mathbb{R}^3$



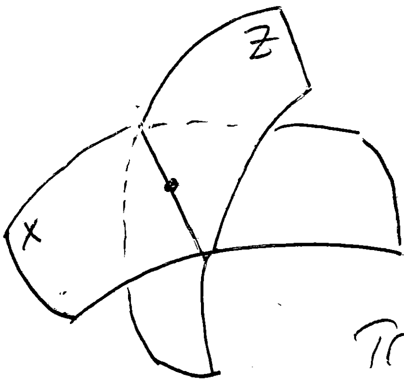
Transverse

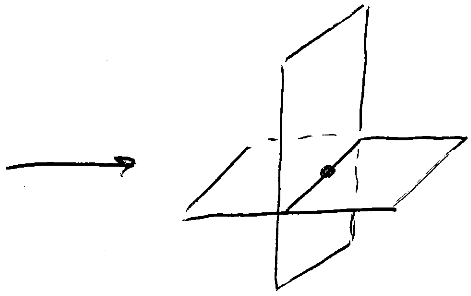




Non-transverse.

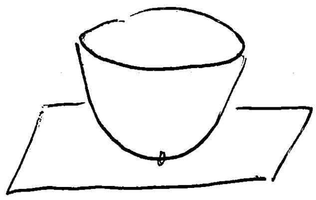
- $\dim X = \dim Y = 2 \qquad Y = \mathbb{R}^3.$

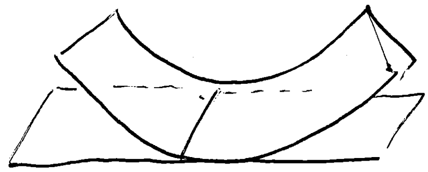


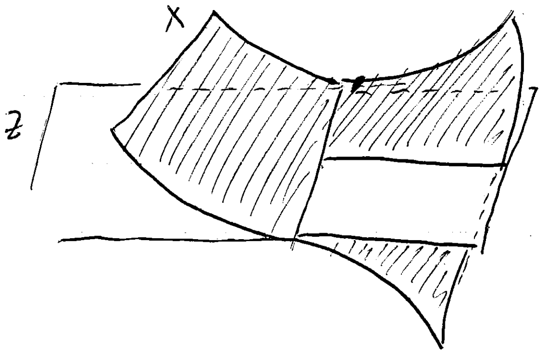


Transverse

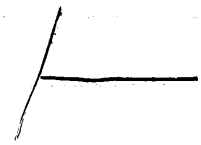
There are many "types" of non-transverse intersections of surfaces. For example:

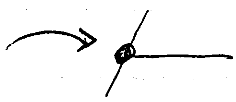




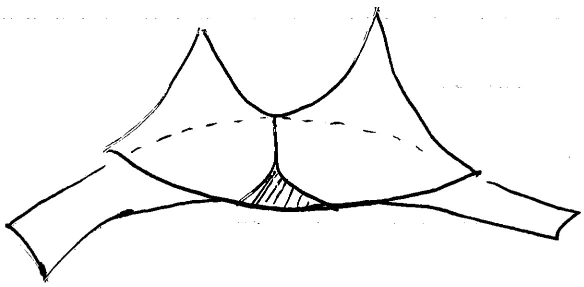


Intersection has T-shape:

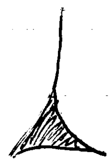


↑ Here, X & Z are not transverse at a single point: → 

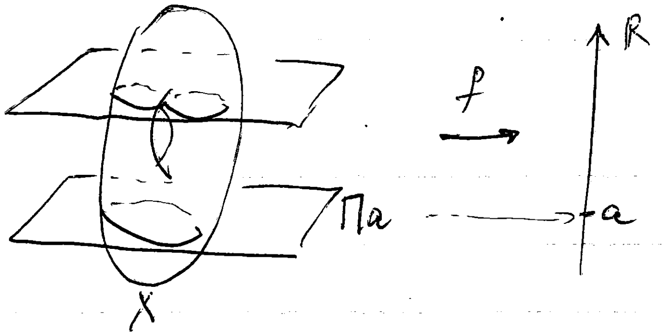
In case $X \cap Z$ non-transverse $X \cap Z$ may not even have "fixed dimension":



Intersection:



Favourite example with torus:



$f : X \to \mathbb{R}$ height func.

Π_a: horizontal plane at height a.

Then $X \cap \Pi_a$ transverse $\Leftrightarrow$ a is a regular value of f
(So: there are 4 non-transverse planes in the picture)

Sard's theorem

Theorem (Sard) If $f: M^m \to N^n$, $m \geq n$, is a smooth map between manifolds, the set of regular values of f is <u>dense</u> in N.

Even stronger, the set of <u>singular</u> (= not regular) values of f has <u>Lebesgue measure 0</u> in N.

<u>Recall</u> A set $S \subset N$ is dense if the closure $\overline{S} = N$, or equivalently ~~$\forall y \in N$ $\exists$ open $U \subset N$, $y \in U$ such that~~

$\forall U \subset N$ open, $U \cap S \neq \emptyset$.

We now recall what it means for a set to have Lebesgue measure 0.

<u>Def</u> Suppose $S \subset \mathbb{R}^n$ is any set. We say S has <u>Lebesgue measure 0</u> if

$\forall \varepsilon > 0$ $\exists$ <u>countable</u> collection $S_i \subset \mathbb{R}^n$ of <u>rectangular solids</u> such that

$$S \subset \bigcup_i S_i \quad \text{and}$$

$$\sum_{i=1}^{\infty} \mathrm{vol}(S_i) < \varepsilon.$$

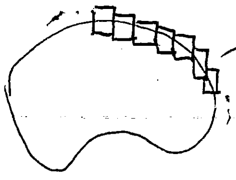
Here, a <u>rectangular solid</u> is simply a product of intervals:

$$[a_1, b_1] \times \ldots \times [a_n, b_n] \subset \mathbb{R}^n$$

and its volume is the usual thing: $(b_1 - a_1) \cdot \ldots \cdot (b_n - a_n) \in \mathbb{R}_+$.

<u>Example</u> • $\{pt\} \subset \mathbb{R}$ has measure 0

• Any curve in $\mathbb{R}^2$ has measure 0.
This is not obvious, but look at example:



— cover the curve by $\varepsilon \times \varepsilon$ "pixels";

number of pixels needed to cover the curve has order <u>$1/\varepsilon$</u>

but the area of each pixel is ε^2.
So total area has order $\varepsilon^2 \cdot 1/\varepsilon = \varepsilon \to 0$

<u>Properties</u>

1 A <u>countable</u> union of measure 0 sets is measure 0

2. If S is measure 0 & $g: \mathbb{R}^n \to \mathbb{R}^n$ is a diffeo, then $g(S)$ is measure 0.

~~K~~ · <u>Note</u> ~~1: is easy because we allow countable collections of S_i in the definition~~

<u>Proof of 1</u>: Cover k'th set by rect-solids of total volume $\frac{\varepsilon}{2^k}$, then $\sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$

<u>2</u>: we skip the proof.

Def

If $S \subset N^n$ is a set in a mfd ~~of~~ dim n, we say S is <u>measure 0</u> if

$\forall y \in N$, $\forall$ parametrization:

$$\begin{array}{c} \text{Nbhood}(y) \subset N \\ \varphi \uparrow \\ U \subset \mathbb{R}^n, \end{array}$$

$\varphi^{-1}(S \cap \text{Nbhood}(y)) \subset \mathbb{R}^n$ is measure 0.

<u>Note</u> By Property 2, we need to check this just for one finite collection of charts covering N.

Property 1 gives a strengthening of Sard's thm:

<u>Thm</u> If $f_i : M \to N$ is a <u>countable</u> family of smooth maps, then the set

$$S = \{ y \in N \mid y \text{ is } \underline{\text{not}} \text{ a reg value for } \underline{\text{some}}\ f_i \} \subset N$$

has <u>measure 0</u>

<u>Proof</u> Follows from Sard + the fact that $\bigcup_{\text{countable}}$ (measure 0 sets) is measure 0.

To prove Sard's theorem, it suffices to prove the following theorem.

<u>Thm</u> (Sard, restated) Let ~~...~~ $f: U \to \mathbb{R}^n$ be a smooth map, where $U \subset \mathbb{R}^m$ is open, and let
$$C = \{x \in U \mid \operatorname{rank} df_x < n\}.$$
Then $f(C)$ has measure 0 in $\mathbb{R}^n$

To see that this is a restatement, note:

- $f(C)$ is precisely the set of critical values
- For $f: M \to N$, cover M by a <u>countable</u> collection of charts (always possible, even if M is non-compact) $U_i \subset M$,

then $$\text{Crit. Values}(f) = \bigcup_{i=1}^{\infty} \text{Crit Values}(f|_{U_i})$$

have measure 0 by theorem above

& use that $\bigcup_{i=1}^{\infty}$ (measure 0 sets) is measure 0

<u>Note</u> We assume that f is of class C^∞ (has all repeated derivatives).

~~Proof~~ Below, we'll write
$f = (f_1, \dots, f_n)$, where $f_i: U \to \mathbb{R}$

Proof Use induction on n, keeping m fixed.
Base: $n=0$ obvious

Recall $C = \{x : \operatorname{rk} df_x < m\} \subset \mathcal{U}$.

Let $C_1 \subset C = \{x : df_x = 0\}$

$\Updownarrow$

all part. deriv. $\left.\frac{\partial f_\ell}{\partial x_1}\right|_x = \dots = \left.\frac{\partial f_\ell}{\partial x_n}\right|_x = 0$ $\underbrace{}_{\text{for all } \ell}$

Let $C_j = \{x \in \mathcal{U} : \left.\partial^j f_\ell / \partial x_{i_1} \dots \partial x_{i_j}\right|_x = 0\}$ $\underbrace{}_{\text{for all } \ell, \text{ all } i_1 \dots i_j}$

(this is the set of pts where all partial derivatives of order $\leq j$ vanish

Then: $C \supset C_1 \supset C_2 \supset \dots\dots$

Proof consists of 3 steps:

① $f(C - C_1)$ has measure 0

② $f(C_i - C_{i+1})$ has measure 0

③ $f(C_k)$ has measure 0 for sufficiently large k

(Clearly, the theorem will follow from these steps: use additivity of measure 0 sets)