Lec 04

Last time:

- Defined local diffeo $f: X \to Y$:

  dim=k

  $d_x f$ must be an iso.
  Stated inverse fcn thm: in a nbhood of x, f has a smooth inverse f^{-1}.

- Defined submersions $f: X \to Y$:

  dim n dim k $n \geq k$

  $d_x f$ must be surjective.
  Proved that f looks like projection:

  $$(x_1 \dots x_k\, x_{k+1} \dots x_n) \to (x_1 \dots x_k)$$

  Submersion thm

  locally in nbhood of x. For this, we extended f locally to F, a map btw nbhoods of same dimension n & used inverse fcn thm.

~~First~~

- Concluded that if $f: X \to Y$, $y \in Y$ and f is a submersion at $\forall$ point $\in f^{-1}(y)$, then $f^{-1}(y)$ is a smooth mfd of dim X − dim Y

  (Such $y \in Y$ is called regular value of f)

  Preimage thm

First let's do ~~the~~ example about $O(n)$.

② $O(n) \subset Mat(n \times n) = \mathbb{R}^{n^2}$ & given by equations:

$$\begin{pmatrix} a_{11} & \dots & a_{n1} \\ \vdots & & \\ a_{1n} & & a_{nn} \end{pmatrix} = A.$$

$$\left.\begin{array}{ll} \sum_j a_{ij}^2 = 1 & \forall i \\ \sum a_{ij} a_{kj} = 0 & \forall i < k \end{array}\right\} \begin{array}{l} n \text{ eqns} \\ \frac{n(n-1)}{2} \text{ eqns} \end{array}$$

Expect: $\dim O(n) = n^2 - n - \frac{n(n-1)}{2} = \frac{n(n-1)}{2}$

~~Lets~~

Let's prove that $O(n)$ is a mfd using Preimage thm. More convenient to do so indirectly (a bit).

Recall that $A \in O(n)$ iff $\underset{\text{transpose}}{AA^t} = \underset{\text{identity}}{I}$

(which is equiv to above eqns).

Now,

$$AA^t \text{ always } \in \underset{\text{space of symmetric matrices}}{S(n)} \cong \mathbb{R}^{n(n+1)/2}$$

So we have a smooth map

$$f: Mat(n) \longrightarrow S(n)$$
$$A \longmapsto AA^t,$$

and $O(n) = f^{-1}(I)$. Must check that I is a reg value of f.

~~As we identify~~ Let's compute, for $A \in \mathrm{Mat}(n)$,

$$\begin{tikzcd} d_A f : & \mathbb{R}^{n^2} \arrow[r] \arrow[d, phantom, "\shortparallel"] & \mathbb{R}^{n(n+1)/2} \\ & \mathrm{Mat}(n) \ni B & \end{tikzcd}$$

By definition,

$$df_A(B) = \lim_{s \to 0} \frac{f(A+sB) - f(A)}{s} =$$

$$= \lim \frac{(A+sB)(A+sB)^t - AA^t}{s} =$$

$$= \lim \frac{1}{s}\left(\cancel{AA^t} + sBA^t + sAB^t + s^2 BB^t - \cancel{AA^t}\right) =$$

$$= \lim \left(BA^t + AB^t + sBB^t\right) =$$

$$= BA^t + AB^t \in \mathrm{Sym}(n) = \mathbb{R}^{n(n+1)/2}$$

We must show: $\forall A \in O(n)$, $d_A f$ is surjective, meaning that:

~~$\forall$ Sym~~ $\forall C \in \mathrm{Sym}(n)$ $\exists B \in \mathrm{Mat}(n)$

solving: $BA^t + AB^t = C$.

<u>Proof</u> $C \in \mathrm{Sym}(n)$ so can write:

$$C = \tfrac{1}{2}C + \tfrac{1}{2}C^t$$

So it suffices to solve $BA^t = \frac{1}{2}C$.

The solution is $B = \frac{1}{2}C(A^t)^{-1} \underset{AA^t = I}{=} \frac{1}{2}CA$

(Here we use $A \in O(n)$, but in fact only need that A is invertible) □

By ~~Submersio~~ Preimage thm, $O(n)$ is a mfd,

$$\dim = n^2 - n(n+1)/2 = n(n-1)/2$$

One more thing about Preimage thm

Suppose $f: X \to Y$, $y \in Y$ regular value
so $f^{-1}(y) = M \subset X$ is a manifold

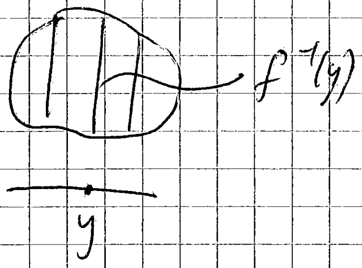
Question For $x \in M$, how do I compute $T_x M \subset T_x X$?

Answer Take $d_x f : T_x X \longrightarrow T_y Y$
$$\cup$$
then $\operatorname{Ker} d_x f = T_x M$

Proof Take parametrisations near x, y
where f looks like:

$$f(x_1 \dots x_n) = (x_1 \dots x_k)$$

& assume x, y corresp to the origin



then $M = f^{-1}(y)$ locally given by $\{x_1 = \dots = x_k = 0\} \times \mathbb{R}^{n-k}$
and here it's clear that
$M = T_0 M = \operatorname{Ker} d_0 f$. □

(Reminder: when I say "f looks like", I'm taking $h = \cancel{\varphi}\, \psi \circ f \circ \varphi^{-1}$ as in prev. lectures. We will frequently skip this point.)

Example Look at $O(n)$. Let $A = I$
then for f: ~~$O(n)$~~ $Mat(n) \longrightarrow Sym(n)$
as above,

$$d_I f(B) = B + B^t$$

and $B \in \operatorname{Ker} d_I f$ iff $B = -B^t$,
i.e. B is skew-symmetric.

(5)

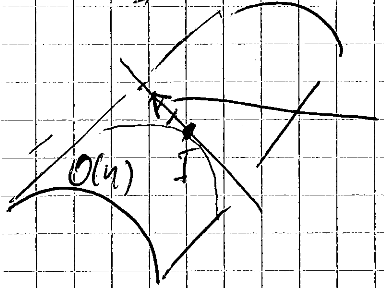
~~Indeed,~~ Conclusion $T_I O(n)$ can be identified (quite canonically) with the space of skew-sym. matrices

Indeed, here's a check:

$$(I+\varepsilon B)(I+\varepsilon B)^t = I+\varepsilon(B+B^t)+\varepsilon^2 BB^t$$

so if B is skew-symmetric,

$$I+\varepsilon B \in O(n) \quad \text{"up to second order"}$$



$I+\varepsilon B$ "stays in $O(n)$" up to second order

Immersions

let $f:$
$$\begin{tikzcd} X \arrow[r] & Y \\ \dim = n \arrow[u] & \dim = k \arrow[u] \end{tikzcd}$$
and now $n<k$

Def We say f is an immersion at $x \in X$ if $d_x f$ is injective.

Local Immersion Thm

If $f: X \to Y$ is an immersion at $x \in Y$, $f(x)=y$, then there are parametrizations φ, ψ such that $h=\psi^{-1}\circ f\circ\varphi$ is given by

$$\underset{\mathbb{R}^n}{(x_1 \dots x_n)} \longmapsto \underset{\mathbb{R}^k}{(x_1 \dots x_n \underbrace{0 \dots 0}_{k-n \text{ zero coordinates}})}$$

Proof Start with any parametrization

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \mathbb{R}^n \supset U \arrow[u, "\varphi"] \arrow[r, "g"'] & V \subset \mathbb{R}^k \arrow[u, "\psi"'] \end{tikzcd}$$

$\varphi(0)=x$

$\psi(0)=y$

$\underline{k>n}$

Know: df_x ~~surjective~~ injective; equivalently dg_0 ~~surj~~ injective

After a linear change of coords on $\mathbb{R}^n$ & $\mathbb{R}^k$, assume

$$dg_0 = \underbrace{\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ \hline & O & \end{pmatrix}}_{n} \Big\} k$$

Now define $G: \mathbb{R}^k \to \mathbb{R}^k$

$$G(x_1 \dots x_n\, x_{n+1} \dots x_k) = g(x_1 \dots x_n) + (0 \dots 0\, x_{n+1} \dots x_k)$$

both are vectors in $\mathbb{R}^k$

Now, $dG_0 : \mathbb{R}^k \to \mathbb{R}^k$ is the identity, by an explicit computation.

$$dG_0 = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right)$$

part coming from dg_0

part coming from $(0 \dots 0\, x_{n+1} \dots x_k)$

Now consider parametrizations φ & $\psi \circ G$:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\varphi"] \arrow[r, "h"'] & V \arrow[u, "\psi \circ G"'] \end{tikzcd}$$

has the canonical form ~~from~~ the statement

Note: $dG_0 = Id \Rightarrow G$ has a smooth inverse in the nbhood of $y \Rightarrow \psi \circ G$ also has a smooth inverse $\Rightarrow \psi \circ G$ is a valid parametrization □

Corollary f is immersion at $x \Rightarrow$ it is an immersion $\forall x' \in \underset{\uparrow}{N}bhood(x)$ (some nbhood)

Corollary f is immersion at x $\Rightarrow$ ~~∄~~ $\exists$ nbhood $x \in U \subset X$ such that $f(U) \subset Y$ is a submfd

Proof

$$\begin{tikzcd} & X \arrow[r, "f"] & Y & \\ \mathbb{R}^n \supset & U \arrow[u, "\varphi"] \arrow[r, "h"] & V \arrow[u, "\psi"'] & \subset \mathbb{R}^k \end{tikzcd}$$

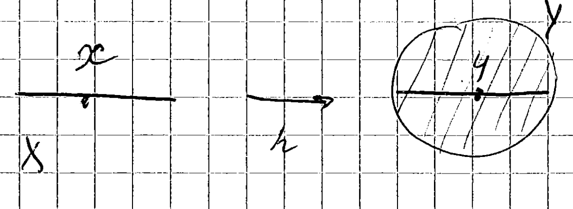
$(x_1 \ldots x_n)$
$\downarrow$
$(x_1 \ldots x_n 0 \ldots 0)$

canonical

then $V \cap \{x_{n+1} = \ldots = x_k = 0\} = V \cap \mathbb{R}^k$

and $\psi|_V$ is a parametrization for $f(\varphi(U))$

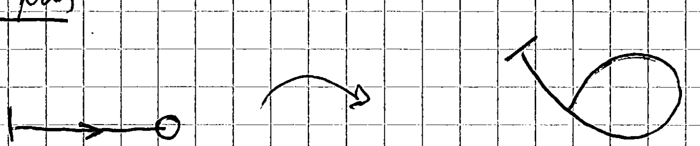
Picture:



Claim If f is an immersion $\forall x \in X$, the image $f(X)$ need not be a submanifold of Y.

Examples

① $X = (-1, 1)$

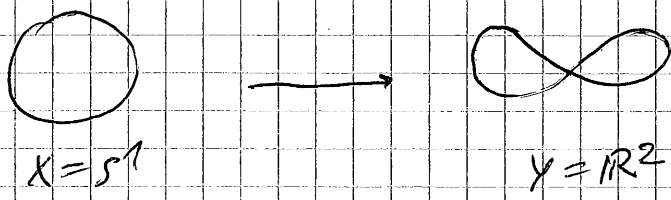


$Y = \mathbb{R}^2$

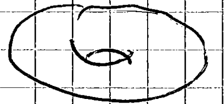
$f(X)$ not a mfd, ~~but this~~

Note Above, f was injective

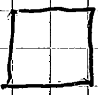
② $f(x)$ not a mfd



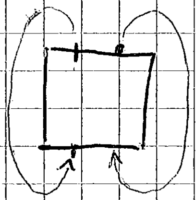
Here, f is not injective (image has a "double point").

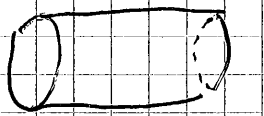
③ Consider the torus:  Notation: $T^2 \cong S^1 \times S^1$

Abstractly, you can obtain it as follows

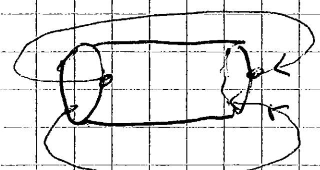
- Start with a square 

- Identify opposite pts on top & bottom

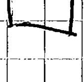


& get a cylinder 

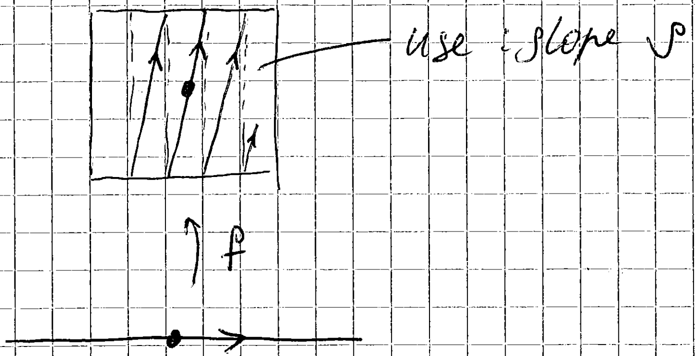
- Identify opposite pts on left & right



& get a torus 

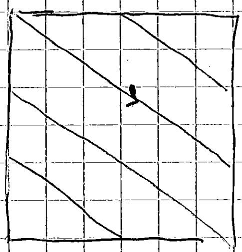
- We can do so simultaneously: ~~[illegible]~~ draw  & think of pairs of opposite pts on ~~the~~ all ~~[illegible]~~ edges identified.

~~Con~~ Pick a slope $s \in$ ~~$\mathbb{Q}\cup\infty$~~ $\mathbb{R}$ & define a map $f: \mathbb{R} \to T^2$ as follows:



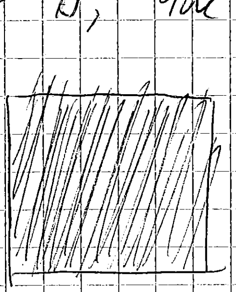
Two possibilities:

(i) $s \in \mathbb{Q}$, then $f(\mathbb{R})$ is a closed circle in T^2, but f is not injective



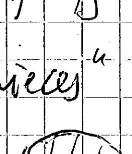
slope $(3, -2) = -\frac{2}{3}$

(ii) $s \notin \mathbb{Q}$, then $f(\mathbb{R})$ covers T^2 densely that is, the closure $\overline{f(\mathbb{R})} = T^2$



("irrational winding")

Then $f(\mathbb{R})$ is not a submfd, because there are ∞ many "pieces" of $f(\mathbb{R})$ passing through any nbhood

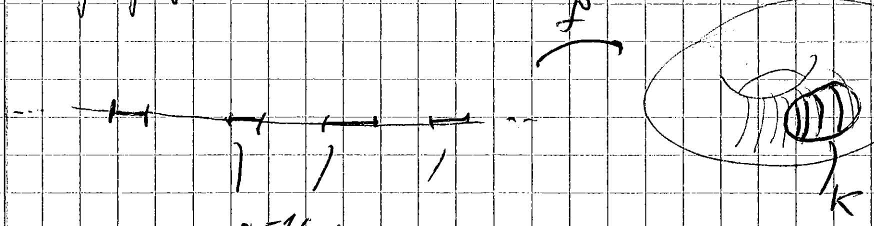


Note: f is injective

The above "bad" examples were due to either: X non-compact, or: f non-injective

Definition $f: X \to Y$ is *proper* if the *preimage* of any compact set is compact.

Example $\mathbb{R} \xrightarrow{f} T^2$ irrational winding is not proper



$f^{-1}(K) = \infty$ many intervals, not compact.

Def $f: X \to Y$ is an *embedding* if it is:

- an immersion (at $\forall x \in X$)
- injective
- proper

Theorem If $f: X \to Y$ is an embedding, then $f(X) \subset Y$ is a submfd, and $f: X \to f(X)$ is a diffeomorphism

Proof It suffices to show that: $\forall$ open $W \subset X$, $f(W)$ is open in Y

(Indeed, take $W(x)$ = nbhood of x, then we saw that $W(x) \xrightarrow{f} f(W(x))$ is a diffeo. If $f(W(x))$ is an *open* nbhood of $f(x)$, this exhibits a chart at $f(x)$).

Suppose $f(W)$ not open, then
$\exists\, y_i \in f(X)$, $y_i \notin f(W)$

But $y_i \xrightarrow[i\to\infty]{} y \in f(W)$

Take the preimages: x_i, x
Observe: $\{y_i\} \cup \{y\} \subset Y$ is compact

$\Rightarrow$ $\{x_i\} \cup \{x\} \subset X$ is compact, by properness of f.
Therefore, passing to a subsequence if necessary, we have

$$x_i \to z \in X;$$

then $f(x_i) \to f(z)$ by continuity.
But also $f(x_i) = y_i \to y$
so $f(z) = y$ so $z = x$, by injectivity of f

By construction, $y \in f(W)$ so $x \in W$
But W is open, using

$$x_i \to x,$$

we have $x_i \in W$ for large enough i.
Then $f(x_i) = y_i \in f(W)$ —''—,
a contradiction □

Note A map $X \xrightarrow{f} Y$ ~~between compact~~ where X is compact
mfld is always proper.
In this case, an embedding $\Leftrightarrow$ an injective immersion.