lec 03

Recall, last time:

- $X \subset \mathbb{R}^n$ k-dim mfd $\rightsquigarrow$ defined $T_x X \subset \mathbb{R}^n$ k-dim vector subspace

- $X \xrightarrow{f} Y$ smooth $\rightsquigarrow$ defined $T_x X \xrightarrow{d_x f} T_{f(x)} Y$.

The definition was:

① extend f to F on a subset open in $\mathbb{R}^n$ & containing x

② compute $d_x F$ using partial derivatives

③ take $d_x F|_{T_x X}$.

We proved $d_x F|_{T_x X}$ only depends on f.
In the proof, we have shown an equiv. definition:

① take parametrisations U of X, V of Y near x resp. y:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \mathbb{R}^{\dim X} \supset U \arrow[u, "\varphi"] & V \subset \mathbb{R}^{\dim Y} \arrow[u, "\psi"'] \end{tikzcd}$$

② let $h: U \to V$ be given by $h = \psi^{-1} \circ f \circ \varphi$, then h is smooth and can compute dh_0 using partial derivatives

(recall: assuming $\varphi(0) = x$).

③ define df_x to be $d\psi_0 \circ dh_0 \circ (d\varphi_0)^{-1}$;

we can take $(d\varphi_0)^{-1}$ because

$$d\varphi_0 : \mathbb{R}^{\dim X} \longrightarrow T_x X \text{ is an } \underline{\text{isomorphism}}$$

by definition of $T_x X$.

<u>General philosophy</u>

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \varphi \uparrow & & \uparrow \psi \\ U & \dashrightarrow & V \end{array}$$

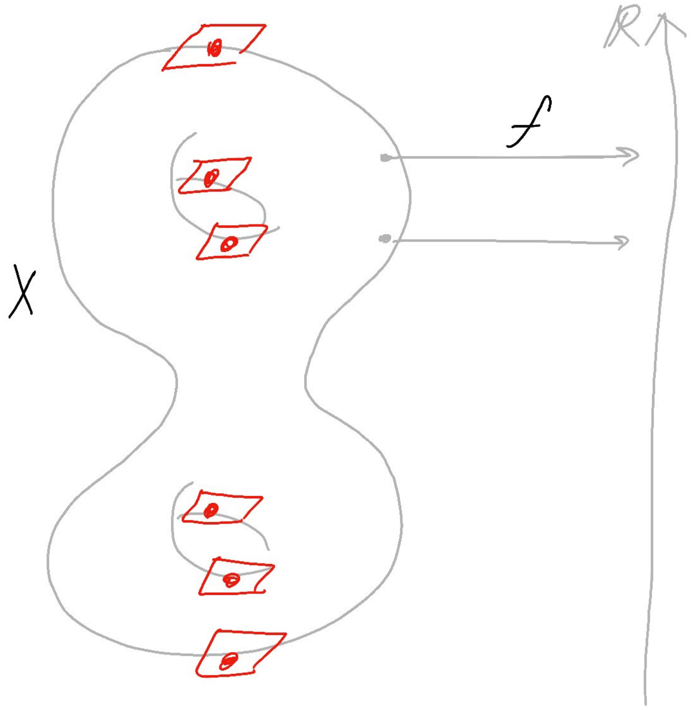
$$h = \psi^{-1} \circ f \circ \varphi$$

To study "local properties" of a smooth map f, it is equivalent to study h, because φ, ψ are diffeomorphisms onto neighbourhoods.

<u>Recall</u> $d_x f$ does <u>not</u> to be injective / surjective / isomorphism in general.

<u>Example</u> $X \subset \mathbb{R}^3$ surface below,

$f : X \longrightarrow \mathbb{R}$ the "height function", i.e. projection to vertical axis.

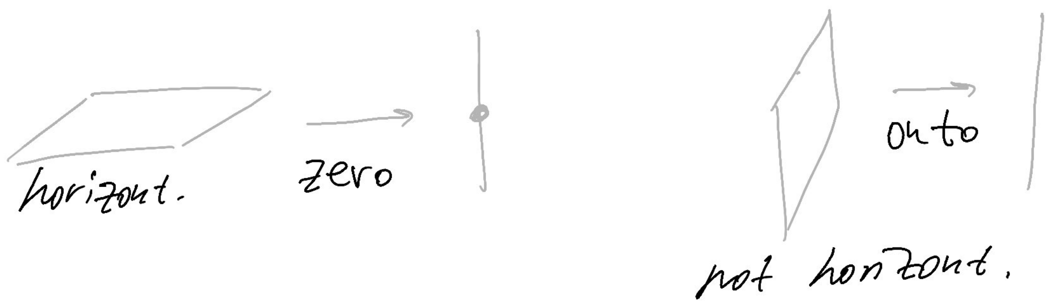


For any $x \in X$ we have $df_x : T_xX \longrightarrow \mathbb{R}$ linear

so there are 2 options:

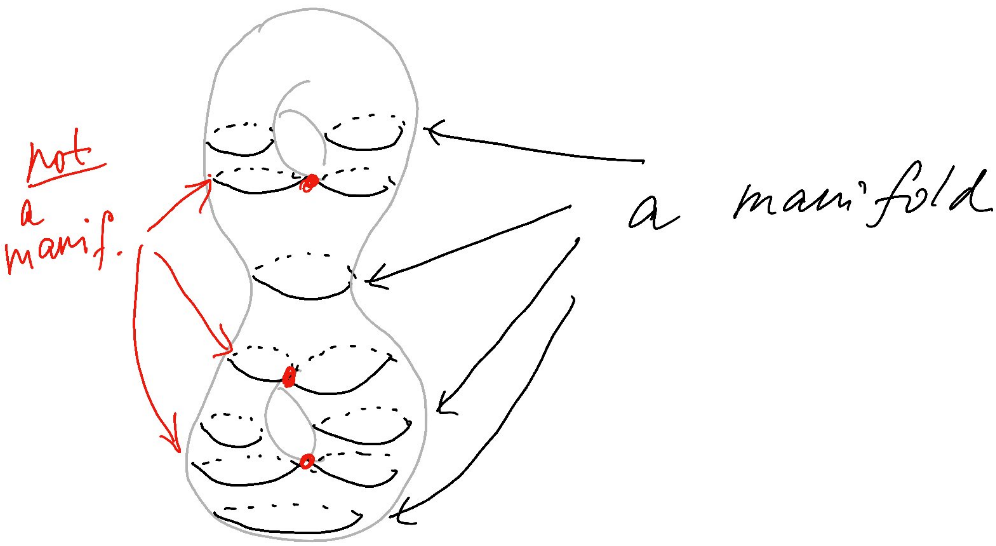
- either df_x is <u>onto</u>
- or df_x is <u>zero</u>

When is df_x zero? In general, $df_x : T_xX \longrightarrow \mathbb{R}$ is the linear projection, so it is zero iff T_xX is <u>horizontal</u>



There are <u>6</u> points in X where T_xX is horizontal, shown above.

Now fix $a \in \mathbb{R}$ and look at the subset $f^{-1}(a) \subset X$, called <u>level set</u>:



We see that:

① If $f^{-1}(a)$ does not contain x s.t. $d_x f = 0$, then $f^{-1}(a)$ is a manifold

② Otherwise $f^{-1}(a)$ is either:

figure 8, not a mfd

point, a mfd but of "wrong dimension".

It turns out that ① is a general theorem

Inverse function thm and submersions

<u>Def</u> Suppose X, Y are mfds and $f : X \to Y$ is smooth. It is called a <u>local diffeomorphism</u> at $x \in X$ if there exist nbhoods

$$\begin{array}{ccc} U \subset X & & V \subset Y \\ \cup & & \cup \\ x & & f(x) \end{array}$$

such that $f|_U$ is a diffeomorphism onto V.

<u>Note</u> $f : X \to Y$ local diffeo

$$\Downarrow$$

$$df_x : T_x X \to T_{f(x)} Y \text{ is an } \underline{\text{isomorphism}}$$

[because $df_x \circ (d f^{-1})_{f(x)} = \mathrm{Id}$, by chain rule]

<u>The inverse function thm</u>

Suppose $f : X \to Y$ smooth map, $x \in X$, and df_x is an isomorphism. Then f is a local diffeo. at x. $\square$

<u>Note</u> Above, we of course must have $\dim X = \dim Y$.

<u>Note</u> This is known from analysis, in the case of $h : \mathbb{R}^n \to \mathbb{R}^n$. Our case translates to the $\mathbb{R}^n$ case by passing to parametrisations, as above
ie take $h = \psi^{-1} \circ f \circ \varphi$

Lemma If $f: X \to Y$ is a local diffeo at $x \in X$, there exist parametrisations φ, ψ such that h is the identity map; equivalently, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\varphi"] \arrow[r, "Id"'] & U \arrow[u, "\psi"'] \end{tikzcd}$$

Proof Take $\psi = f \circ \varphi$, then it is a parametrisation because it have a smooth inverse:

$$\varphi^{-1} \circ f^{-1}$$

(defined on nbhood of y). $\square$

Before, we had $\dim X = \dim Y$.
Now assume $\dim X \geq \dim Y$.

Def A smooth map $f: X \to Y$ is called a submersion at $x \in X$ if df_x is surjective

$\dim X = n \geq \dim Y = k$

Local submersion thm Suppose $f: X \to Y$ submersion at x; $y = f(x)$.
Then there exist local parametrisations φ, ψ such that $h = \psi^{-1} \circ f \circ \varphi$ is given by:

$$h(x_1, \dots, x_k, x_{k+1}, \dots x_n) = (x_1 \dots x_k) \in \mathbb{R}^k$$

"projecting to the first k coordinates"

Here's the diagram:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \mathcal{U} \arrow[u, "\varphi"] \arrow[r, "h"'] & V \arrow[u, "\psi"'] \end{tikzcd}$$

$\mathcal{U} \subset \mathbb{R}^n$, $V \subset \mathbb{R}^k$

h = projection to first k coord.

<u>Proof</u> Start with <u>some</u> parametrisations φ, ψ, and let $g = \psi^{-1} \circ f \circ \varphi$.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \mathbb{R}^n \supset \mathcal{U} \arrow[u, "\varphi"] \arrow[r, "g"] & V \subset \mathbb{R}^k \arrow[u, "\psi"'] \end{tikzcd}$$

$\varphi(0) = x$
$\psi(0) = y$

f immersion $\Rightarrow$ dg_0 is surjective.
By linear algebra, after a linear change of coords we can bring dg_0 to the form:

$$k \left\{ \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & & & \vdots & & \\ \vdots & & \ddots & & & & \\ 0 & & & 1 & 0 & \dots & 0 \end{pmatrix} \right. \qquad \underbrace{}_{k} \; \underbrace{}_{n-k}$$

$dg_0 : \mathbb{R}^n \longrightarrow \mathbb{R}^k$

Now define $G : \mathcal{U} \to \mathbb{R}^n$ by:

$$G(x_1,\dots,x_n) = \Big(\underbrace{g(x_1,\dots,x_n)}_{\in\,\mathbb{R}^k},\ \underbrace{x_{k+1},\dots,x_n}_{\in\,\mathbb{R}^{n-k}}\Big) \in \mathbb{R}^n$$

Then dG_0 is the <u>identity</u> $n\times n$ matrix

So G_0 is a <u>local diffeomorphism</u>, by inverse fcn thm.

It has an inverse G_0^{-1}, defined on a nbhood of 0

Consider another parametrisation:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\varphi\circ G^{-1}"] \arrow[r, "h"] & V \arrow[u, "\psi"'] \end{tikzcd}$$

new parametrisation

maybe smaller as previously

<u>Claim</u> h is now the standard projection, as required.

<u>Let's check</u>: We have

$h = g\circ G^{-1}$ by construction, or equivalently:

$h\circ G = g$. It suffices to see that

[standard projection] $\circ\, G = g$.

Indeed:

$$G = (\underbrace{g(X_1 \dots X_n),\ X_{k+1} \dots X_n}_{\text{forgets these coords}})$$

$$\Big\downarrow \text{std. proj.}$$

$$g(X_1 \dots X_n).$$

We proved the claim, hence the thm. $\square$

Def Let $f: X \to Y$ smooth map.
A point $y \in Y$ is called <u>regular value</u> of f if

$\forall x$ s.t. $\underline{f(x) = y}$, df_x is <u>surjective</u>

Def We call $\{x : f(x) = y\}$ the <u>preimage</u>, denoted by $f^{-1}(y)$.

<u>Preimage theorem</u> If $y \in Y$ is a regular value of f, then the preimage

$f^{-1}(y)$ is a smooth mfd.

of dimension $\dim X - \dim Y$.

<u>Proof</u> Use Submersion theorem to find local param such that

$$h = \varphi \circ f \circ \psi^{-1} : \underset{\mathbb{R}^n}{\mathcal{U}} \to \underset{\mathbb{R}^k}{V} \quad \text{equals:}$$

$$h(X_1 \dots X_k\ X_{k+1} \dots X_n) = (X_1 \dots X_k) \in \mathbb{R}^k$$

The preimage of $0\in\mathbb{R}^k$, $h^{-1}(0)$, equals:

$$\begin{cases} x_1 = 0 \\ \vdots \\ x_k = 0 \end{cases}$$

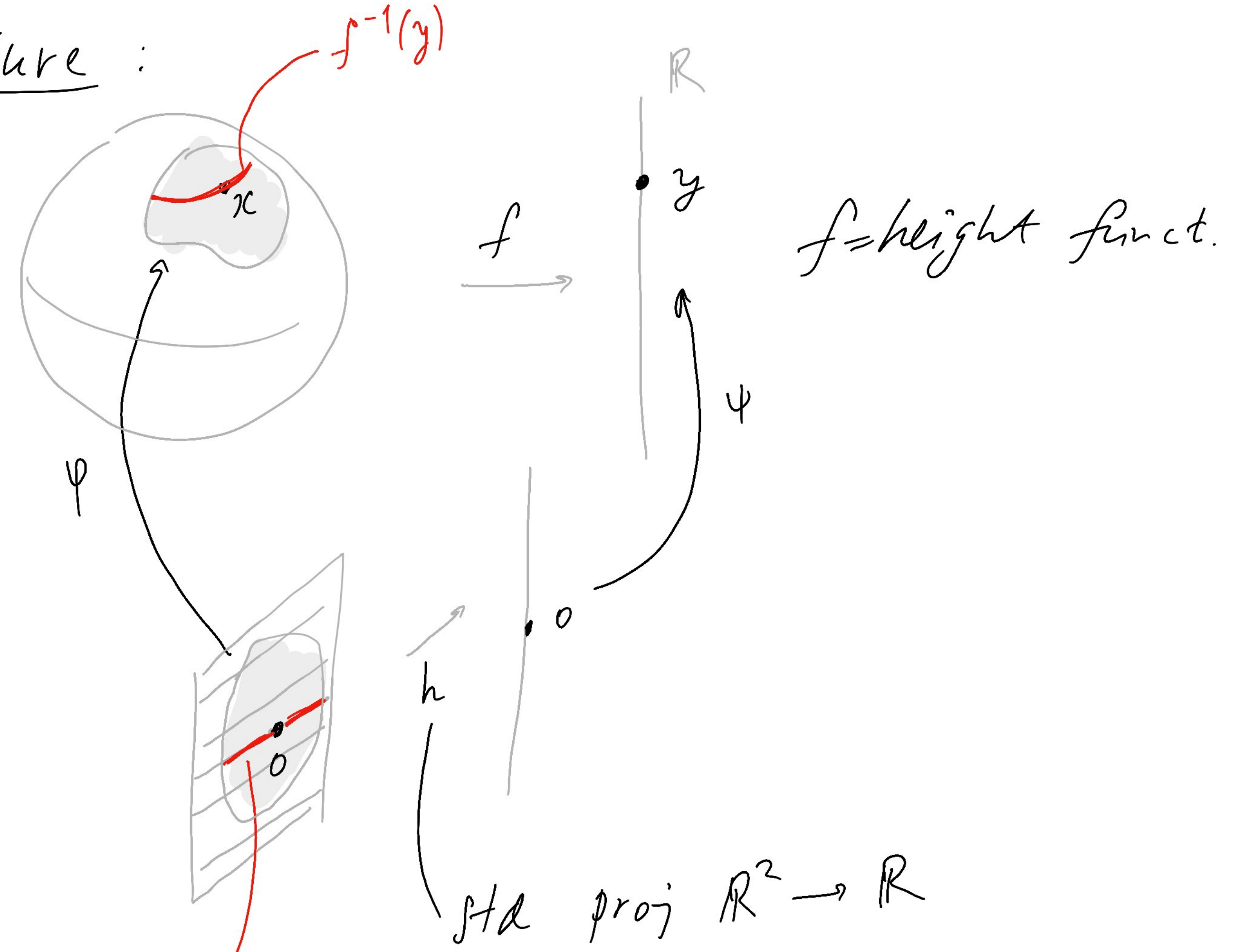
So $U\cap\underbrace{\{x_1=0 \dots x_k=0\}}_{\mathbb{R}^{n-k}} \subset \mathbb{R}^{n-k}$ is a chart

for $f^{-1}(y)$:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\varphi"] \arrow[r, "h"'] & V \arrow[u, "\psi"'] \end{tikzcd} \qquad \begin{tikzcd} f^{-1}(y) \arrow[r] & Y \\ U\cap\mathbb{R}^{n-k} \arrow[u, "\varphi|_{U\cap\mathbb{R}^{n-k}}"] \arrow[r, "0"'] & V \arrow[u, "\psi"'] \end{tikzcd} \quad \psi(0)=y$$

std. projection

<u>Picture</u> :



$h^{-1}(0)$ is a piece of straight line.
This is a chart for $f^{-1}(y)$.

Example Lets prove that the unit sphere

$$S^n \subset \mathbb{R}^{n+1}$$

is a manifold. Consider

$$f : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$$

$$f(x_1 \dots x_{n+1}) = \sum_{i=1}^{n+1} x_i^2$$

then $S^n = f^{-1}(1)$.

To prove it's a mfd, it suffices to check that 1 is a regular value for f.

Compute $df_x = (\partial f/\partial x_1 \dots \partial f/\partial x_{n+1})_x =$

$$= (2x_1, \dots, 2x_{n+1}).$$

This can be seen as the "matrix" of $df_x : \mathbb{R}^{n+1} \to \mathbb{R}$. In this case:

df_x is surjective $\iff$ df_x is nonzero $\iff$ $x \neq 0 \in \mathbb{R}^{n+1}$

The preimage $f^{-1}(1)$ does not contain $0 \in \mathbb{R}^{n+1}$ So 1 is a regular value of f.

So $S^{n+1} = f^{-1}(1)$ is a manifold.

More generally,

<u>Corollary</u> (of the Submersion thm):

Suppose $g_1 \dots g_k : \mathbb{R}^n \to \mathbb{R}$ smooth functions; and $X \subset \mathbb{R}^n$ is defined by a system of equations:

$$\begin{cases} g_1(x_1 \dots x_n) = 0 \\ \vdots \\ g_k(x_1 \dots x_n) = 0 \end{cases}$$

Suppose $x \in X$. Consider the derivatives:

$$(dg_i)_x : \mathbb{R}^n \to \mathbb{R}$$

or equivalently:

$$(dg_i)_x \in \underbrace{(\mathbb{R}^n)^*}_{\text{dual vector space}} \cong \mathbb{R}^n$$

Then X is a smooth manifold of $\dim X = n-k$ if $\forall x \in X$, the vectors

$$\{(dg_i)_x\}_{i=1 \dots k} \in (\mathbb{R}^n)^*$$

are linearly independent.

In practice, to check this means to check that the following $n \times k$ matrix has full rank k:

$$\begin{pmatrix} \partial g_1(x)/\partial x_1 & \cdots & \partial g_1(x)/\partial x_n \\ \vdots & & \\ \partial g_k(x)/\partial x_1 & \cdots & \partial g_k(x)/\partial x_n \end{pmatrix}$$

Proof Consider the map

$$g = (g_1 \dots g_k) : \mathbb{R}^n \longrightarrow \mathbb{R}^k$$

Then $X = g^{-1}(0)$, $0 \in \mathbb{R}^k$.

If the above matrix has full rank everywhere, this means dg_x is surjective everywhere

$\Rightarrow g$ is a submersion. By Submersion thm,

X is a mfd. $\square$