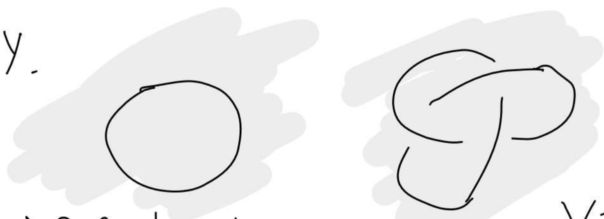
Lec 02 Another example to diffeomorphisms.
This is an example when $X \subset \mathbb{R}^3$, $Y \subset \mathbb{R}^3$ are diffeomorphic, but there exists no global diffeomorphism $F: \mathbb{R}^3 \to \mathbb{R}^3$ such that $F(X) = Y$.



X = trivial knot

Y = some non-trivial knot

Definition of a manifold

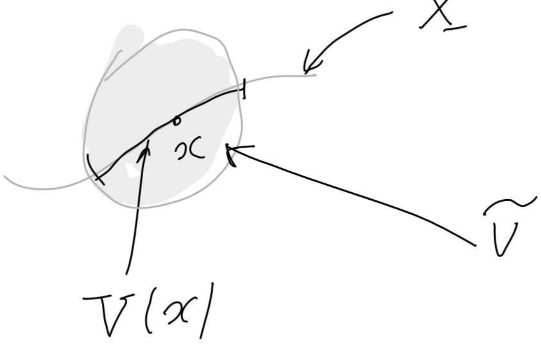
Def A subset $X \subset \mathbb{R}^k$ is an n-dimensional mfd if

$\forall x \in X$, $\exists$ nbhood $V(x) \subset X$

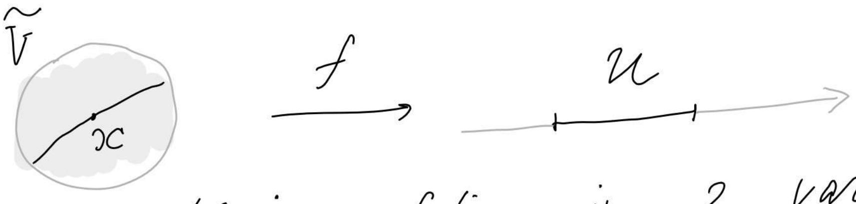
such that there is a diffeomorphism

$f: V(x) \to$ an open subset $U \subseteq \mathbb{R}^n$.

- Recall that this means in particular that f is defined on an open set $\tilde{V} \subset \mathbb{R}^k$ containing $V(x)$:



Eg:



(f is a fction in 2 variables).

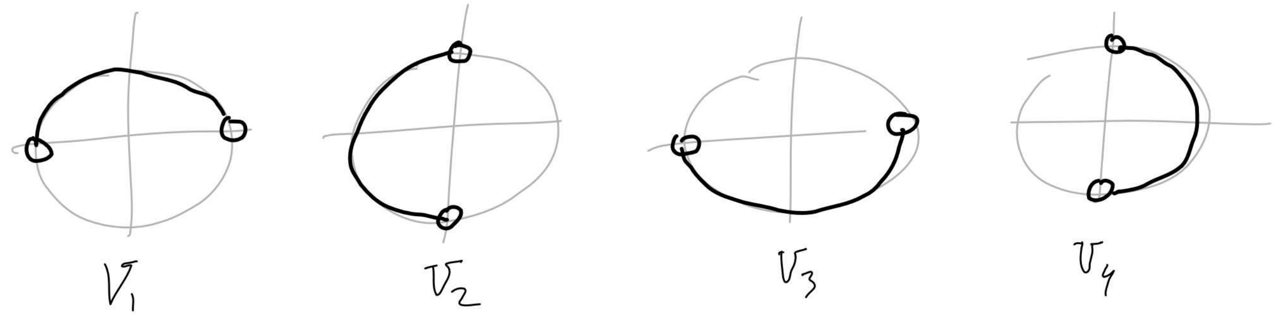
- Observe that f has to be bijective only when restricted to $V(x) \subset X$.
- Remember that $f^{-1} : U \to V(x)$ must also be a smooth map, which by definition means that f^{-1} is a smooth map $U \to \mathbb{R}^k$ whose image lands in $V(x)$.

<u>Some terminology</u>

- A set $V(x)$ from above is called a <u>chart</u>
- The map $\varphi = f^{-1} : U \longrightarrow V(x)$ is called a <u>parametrisation</u> of the chart $V(x)$, or <u>local coordinates</u> on $V(x)$. We usually assume $\varphi(0) = x$.

<u>Example</u> $X = \{x^2+y^2=1\} \subset \mathbb{R}^2$ (circle) is a 1-dim'l mfd.

<u>Proof</u>. We cover X by 4 charts: $X = \bigcup_{i=1}^{4} V_i$ where V_i are as shown:



For each V_i, $\exists$ diffeomorphism

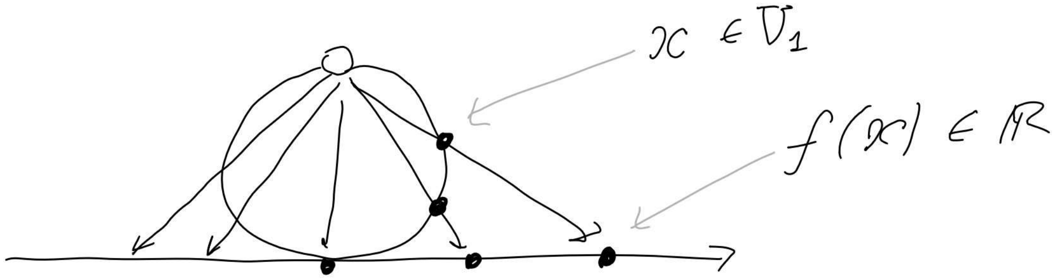
$$V_i \longrightarrow U = (-1, 1) \subset \mathbb{R}$$

as we have previously seen.
This proves X is a manifold: indeed, $\forall$ point $x \in X$, take a $V_i \ni x$ and use it as a chart $V(x)$. $\square$

<u>Note</u> One can cover X by <u>two</u> charts, but never by <u>one</u> chart (as proved in the earlier examples).
To cover X by two charts, consider

$$V_1 = X \setminus \{p_1\} \qquad V_2 = X \setminus \{p_2\}$$

& take diffeomorphism $f : V_1 \longrightarrow \mathbb{R}$ which is the <u>stereographic projection</u>:



One must write f by a formula to verify it's a diffeomorphism.

Example Similarly,

$$S^n = \{ \sum_{i=1}^{n+1} X_i^2 = 1 \} \subset \mathbb{R}^{n+1}(X_1 \dots X_{n+1})$$

is an n-dim'l mfd, and can be covered by 2 charts

Example (Product)

$X \subset \mathbb{R}^{k_1}$, $Y \subset \mathbb{R}^{k_2}$ mfds

then $X \times Y \subset \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$ is also a mfd, and $\dim(X \times Y) = \dim X + \dim Y$.

Proof Let $x \in X$, $y \in Y$, and find charts $V(x) \subset X$, $W(y) \subset Y$

& diffeomorphisms

$$f_1 : V(x) \to U_1 \subset \mathbb{R}^{\dim X}$$

$$f_2 : W(y) \to U_2 \subset \mathbb{R}^{\dim Y}$$

Then $V(x) \times W(y) \subset X \times Y$ is a chart for $X \times Y$, with a diffeo.

$$f_1 \times f_2 : V(x) \times W(y) \to U_1 \times U_2$$

defined by $(f_1 \times f_2)(v, w) = (f_1(v), f_2(w))$

$v \in V(x)$
$w \in W(y)$.

- $U_1 \times U_2$ is open in $\mathbb{R}^{\dim X + \dim Y}$
- $f_1 \times f_2$ extends smoothly on a product open nbhood of $V(x) \times W(y)$, is bijective, and has smooth inverse $\square$

Derivatives & tangents

Let $U \subset \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}^k$ be a smooth map. One can write it in coordinates:

$$f = \begin{pmatrix} f_1(x_1 \dots x_n) \\ \vdots \\ f_k(x_1 \dots x_n) \end{pmatrix}$$

where $f_i : U \to \mathbb{R}$ and $x_1 \dots x_n$ are the coords. on $\mathbb{R}^n$ (and on U).

Recall: for $h \in \mathbb{R}^n$ and $x \in U$,

$$df_x(h) = \lim_{t \to 0} \frac{f(x+th) - f(x)}{t} \in \mathbb{R}^k.$$

Fix x & let h vary, then we get

$$df_x : \mathbb{R}^n \longrightarrow \mathbb{R}^k.$$

a <u>linear</u> map.

<u>Note</u> this linear map is given by the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & & \\ \frac{\partial f_k}{\partial x_1}(x) & \cdots & \frac{\partial f_k}{\partial x_n}(x) \end{pmatrix}$$

$n \times k$ matrix.

<u>Recall</u> Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ open, and let f, g be smooth maps between

$$\begin{tikzcd} x \in U \arrow[r, "f"] \arrow[rr, bend right, dashed, "g \circ f"'] & V \arrow[r, "g"] & \mathbb{R}^k \end{tikzcd}$$

Then the <u>chain rule</u> says:

$$d(g \circ f)_x = (dg)_{f(x)} \circ df(x)$$

$\forall x \in U$

Here:
$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "df_x"] \arrow[rr, bend right, dashed, "d(g \circ f)_x"'] & \mathbb{R}^m \arrow[r, "(dg)_{f(x)}"] & \mathbb{R}^k \end{tikzcd}$$

Recall df_x is the best linear approximation of f at $x \in \mathbb{R}^n$:

$$f(x+h) = f(x) + df_x(h) + \bar{o}(\|h\|)$$

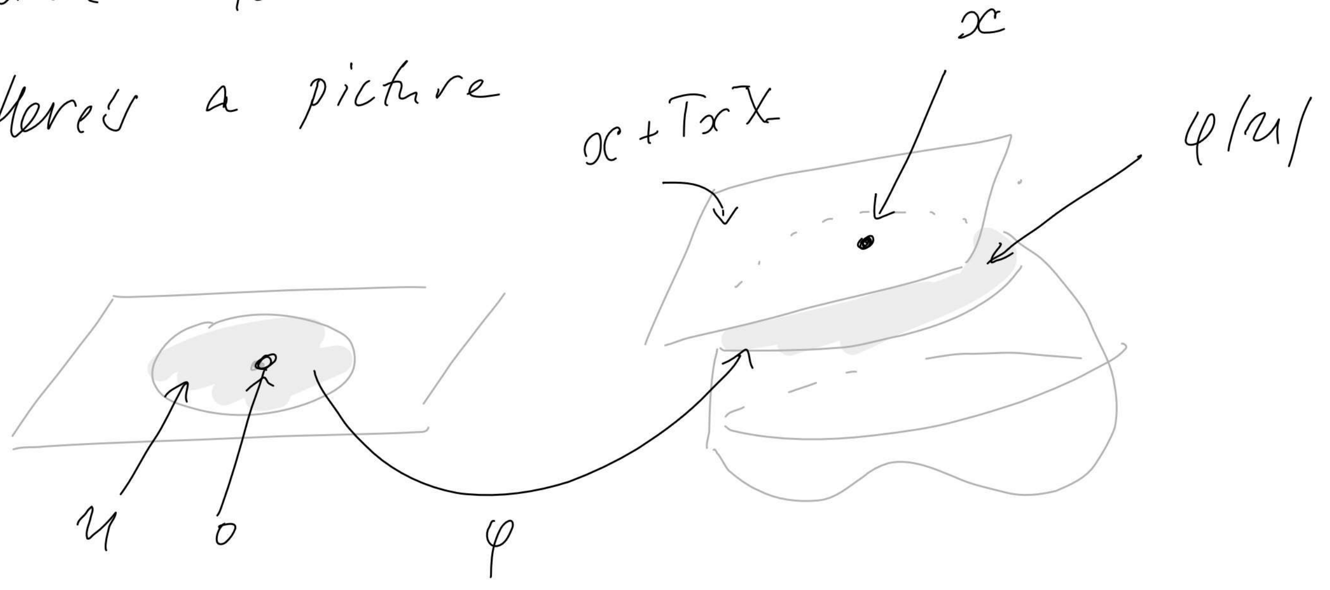
Def (Tangent space to a mfd)
Let $X \subset \mathbb{R}^n$ be a k-dim'l mfd and $x \in X$ a point.
The tangent space $T_x X \subset \mathbb{R}^n$ is a k-dim'l subspace defined as follows.

Let $\varphi: U \to X$ be a local parametrisation near x, $\varphi(0) \in x$,

$$T_x X = \text{image of } d\varphi_0$$

where $d\varphi_0 : \mathbb{R}^k \to \mathbb{R}^n$.

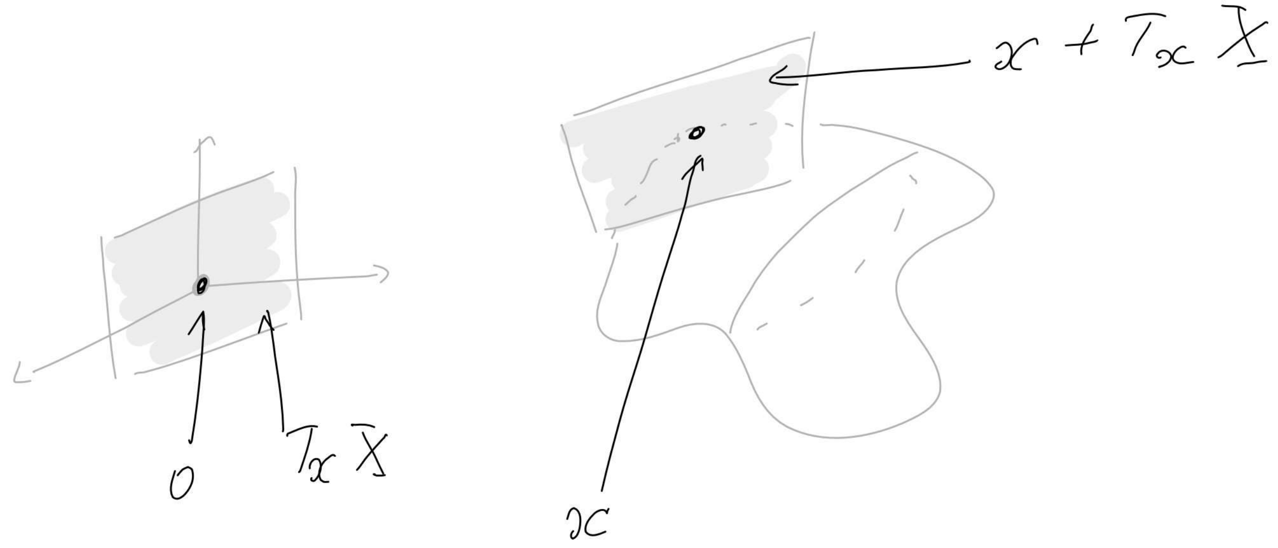
Here's a picture



A small caveat/annoying detail: according to our def'n, $T_x X$ is always a plane passing through the origin.

The actual tangent plane we think of geometrically is the translate of $T_x X$:

$$x + T_x X$$



<u>Lemma</u> $T_x X \subset \mathbb{R}^n$ does not depend on a choice of local parametrisation $\varphi: U \to X$.

Suppose $\varphi: U \to X$ are two pairs.
$\psi: V \to X$

We have $\varphi(0) = \psi(0) = x$ & can assume $\varphi(U) = \psi(V)$.
(otherwise redefine $U = \varphi^{-1}(\varphi(U) \cap \psi(U))$
$V = \psi^{-1}(\varphi(U) \cap \psi(U))$)

Then $h = \psi^{-1}\varphi: U \to V$ is a diffeo, because it's a composition of diffeos
(Exercise: f, g diffeos $\Rightarrow f \circ g$ diffeo, f^{-1} diffeo)

Write $\varphi = \psi \circ h$, then:

$$d\varphi_0 = d\psi_0 \circ dh_0$$

$\Rightarrow$ Image $(d\varphi_0) \subseteq$ Image $(d\psi_0)$.

The converse is analogous. $\square$

<u>Lemma</u> $\dim T_x X = \dim X \ (=k)$,
and $d\varphi_0 : \mathbb{R}^k \to T_x X$ is an isomorphism,
for any loc. param. φ.

<u>Proof</u> Let $\varphi : \underset{\mathbb{R}^k}{\overset{\cap \text{open}}{U}} \to X$ be a param. at x

and $\varphi^{-1} : X \to U$ the inverse.

φ^{-1} is smooth meaning:

$\exists \underset{x}{\overset{\text{open}}{W}} \subset \mathbb{R}^n$, $\exists F : W \to \mathbb{R}^k$ st

$$F \circ \varphi = \mathrm{Id}.$$

Derivate this:

$$\begin{tikzcd} \mathbb{R}^k \arrow[r, two heads, "d\varphi_0"] \arrow[rr, bend right, dashed, "\mathrm{Id}"'] & T_x(X) \arrow[r, "dF_x"] & \mathbb{R}^k \end{tikzcd}$$

The composition is Id, and $d\varphi_0$ is surjective (by def. of $T_x X$)

$\Rightarrow$ both arrows are isomorphisms $\square$

Def (Derivative of a smooth map)
(sometimes called differential)

Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ be mfds;
$f : X \to Y$ smooth map (not nec. diffeo'm)
$x \in X$, $y \in Y$, $f(x) = y$.

The derivative is a linear map

$$df_x : T_x X \to T_y Y$$

defined as follows.
Take any smooth map

$$F : W \longrightarrow \mathbb{R}^m,$$

$$x \in W \subset \mathbb{R}^n \quad \text{open},$$

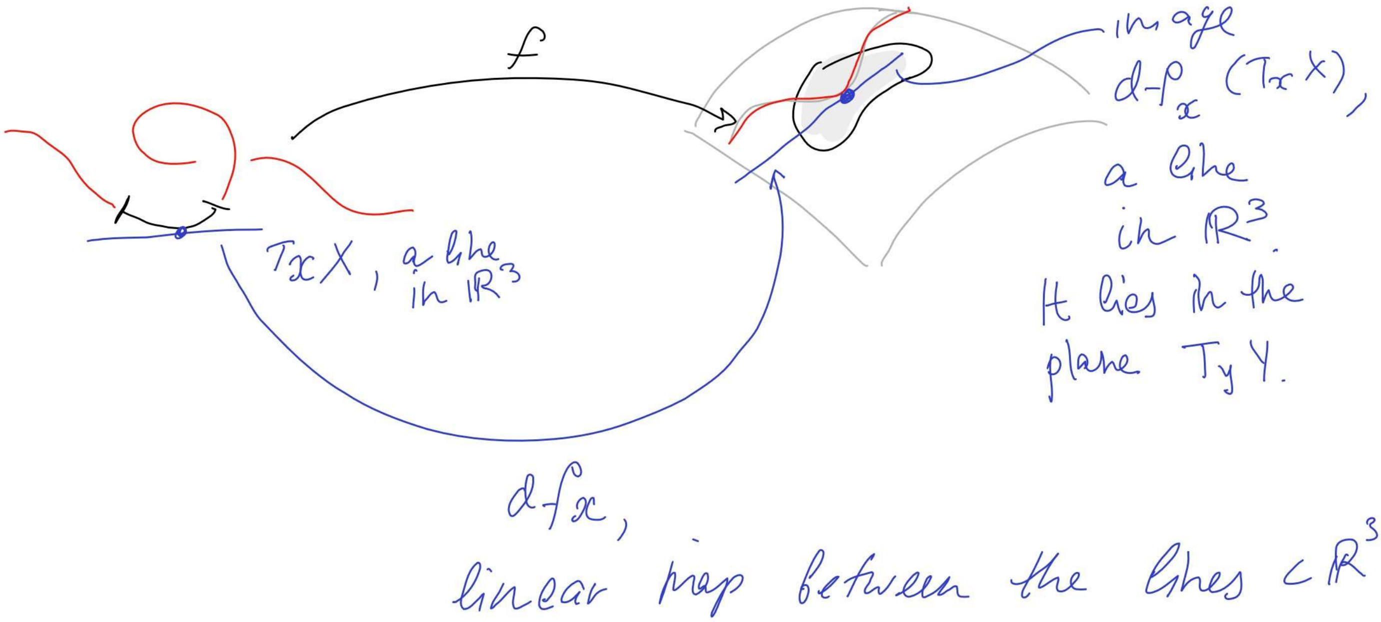
extending f in nbhood $W \cap X$ of x
For $v \in T_x X \subset \mathbb{R}^n$

$$df_x(v) \stackrel{def}{=} dF_x(v),$$

where $dF_x : \mathbb{R}^n \to \mathbb{R}^m$ is the derivative in the classical analysis sense:
it's given by partial derivatives

$$\left(\partial F^i(x) / \partial x^j \right) \quad \begin{matrix} i = 1 \ldots m \\ j = 1 \ldots n \end{matrix}.$$

Here's a picture:



<u>Lemma</u> df_x does not depend on the choice of extension F.

<u>Proof</u>. Consider local parametrisations at x, y
$ \Pi \quad \Pi$
$ X \quad Y$

$$\begin{array}{ccc} X & & Y \\ \varphi \uparrow & & \downarrow \psi^{-1} \\ \mathbb{R}^{\dim X} \underset{open}{\supseteq} U & & V \underset{open}{\subset} \mathbb{R}^{\dim Y} \end{array}$$

extends to a smooth function on a nbhood of y

Take $x \in W \subset \mathbb{R}^n$ open as in the above def'n; assume U is small enough so that $\varphi(U)$ lands in W, and that $F(W)$ lands an open nbhood Z y on which ψ^{-1} extends smoothly.

Rewrite the above maps φ, ψ, adding F defined on W:

$$\begin{tikzcd} \mathbb{R}^k \underset{\text{open}}{\supseteq} W \arrow[r, "F"] & Z \overset{\text{open}}{\subseteq} \mathbb{R}^m \arrow[d, "\psi^{-1} \text{ smooth}"] \\ \underset{\text{open}}{U} \arrow[u, "\varphi"] \arrow[r, dashed, "h"'] & \underset{\text{open}}{V} \end{tikzcd}$$

Denote $h = \psi^{-1} \circ F \circ \varphi$, note it is smooth

Next, we actually have
$$h = \psi^{-1} \circ f \circ \varphi$$
because $\operatorname{Im} \varphi \subset X$ and $F|_{W \cap X} \equiv f$.

Consider derivatives of functions between open sets:

$$\begin{tikzcd} \mathbb{R}^k \arrow[r, "dF_x"] & \mathbb{R}^m \\ \mathbb{R}^{\dim X} \arrow[u, "d\varphi_0"] \arrow[r, "dh_0"'] & \mathbb{R}^{\dim Y} \arrow[u, "d\psi_0"'] \end{tikzcd}$$

The diagram commutes, so

dF_x takes $\underbrace{\operatorname{Image}(d\varphi_0)}_{= \text{def } T_x X}$ inside $\underbrace{\operatorname{Image}(d\psi_0)}_{= \text{def } T_y Y}$

So $dF_x\big|_{T_xX}$ is a map $T_xX \to T_yY$.

Also, we can write $dF_x|_{T_xX}$ as:

(*) $d\psi_0 \circ dh_0 \circ (d\varphi_0)^{-1}$

using that $d\varphi_0 : \mathbb{R}^{\dim X} \to \mathbb{R}^k$ is an isomorphism <u>onto</u> T_xX (see above), therefore has an inverse

$$(d\varphi_0)^{-1} : T_xX \to \mathbb{R}^{\dim X}.$$

Finally, (*) does not depend on F, since h does not depend on F (only on f).

$\square$

<u>Note</u> It may seem at the end of proof that df_x depends on the parametrisations, but we know that it doesn't, by the original definition!

<u>Lemma</u> Image $(df_x) \subset T_yY$

<u>Proof</u> Shown above $\square$

<u>Lemma</u> We have chain rule. If

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \Psi & & \Psi & & \Psi \\ x & \longmapsto & y & \longmapsto & z \end{array}$$

are smooth maps then

the diagram commutes:

$$\begin{tikzcd} T_x X \arrow[r, "df_x"] \arrow[rr, bend right, "d(g\circ f)_x"'] & T_y Y \arrow[r, "dg_y"] & T_z Z \end{tikzcd}$$