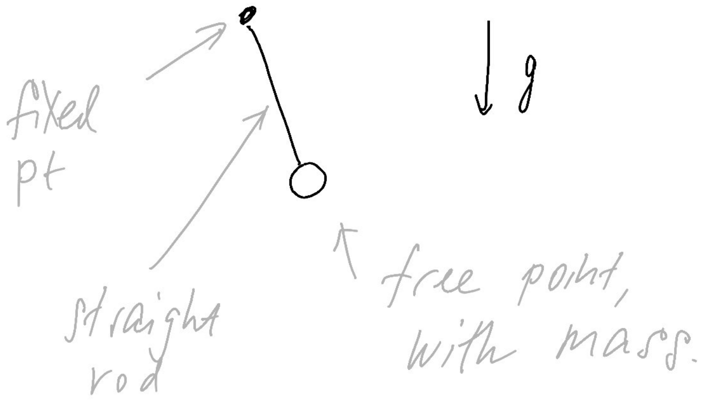
Lec 01

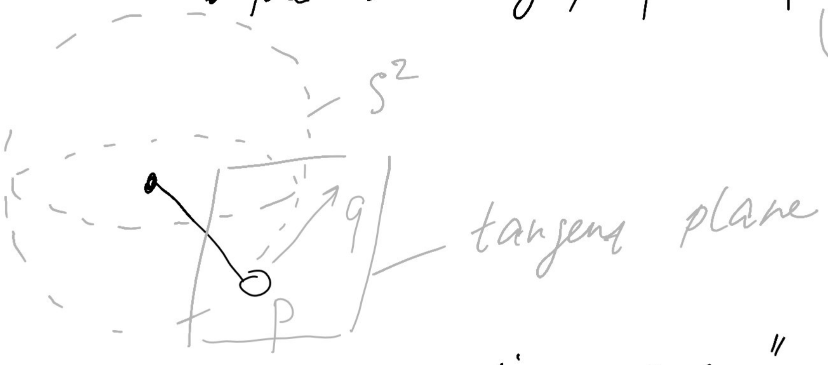
Intro Consider a classic mechanical system: the pendulum.



The endpoint rotates freely in $\mathbb{R}^3$, subject to gravity. To specify the system at a given moment of time, one must know:

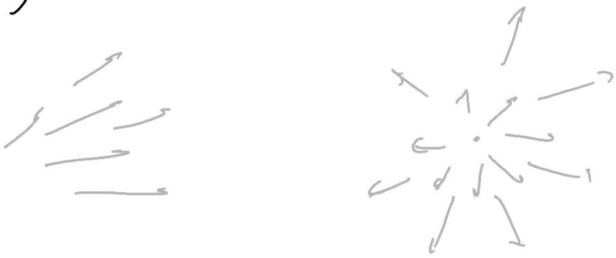
- the position of endpoint, $p \in \underbrace{S^2}_{\text{unit sphere}} \subset \mathbb{R}^3$

- the velocity, $q \in T_p S^2$ "tangent plane" to S^2 at p



The "configuration space" of the system is therefore the union of the tangent planes to the sphere.

This is an example of a *manifold*, and the evolution of the system is given by the flow of a *vector field* on it.

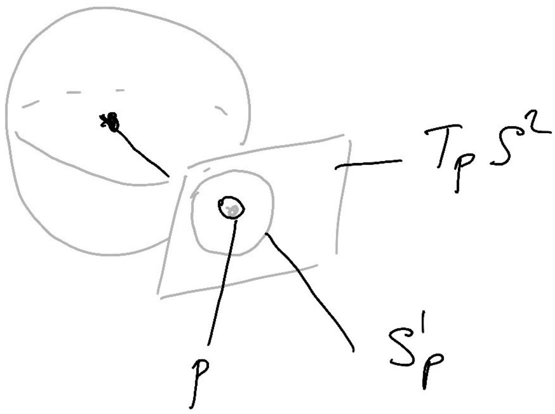


examples of vector field.

In fact, the config. space of the pendulum can be reduced to

$$M = \bigcup_{p \in S^2} S^1_p$$

where $S^1_p \subset T_p S^2$ is the unit circle in the tangent plane.



It turns out that M is *diffeomorphic* to S^3, the 3-sphere.

The vector field on S^3 defining the pendulum motion has two *equilibria*, ie points where the vector fields is zero

Can there be a vector field on S^3
(eg another mechanical system on S^3)
with <u>no</u> equilibria?

<u>Answer</u>: Yes.

<u>But</u>: For some other manifolds, No!

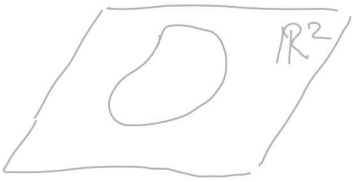
For example, any vector field on S^2
must have an equilibrium.

The answer to this question depends on
the <u>topology</u> (<u>ie</u> global structure)
of the manifold.

Intuitively, a manifold X is a set / topological space which locally (ie near each pt of X) looks like a piece of $\mathbb{R}^n$, where k is called the dimension, $\dim X$.

There are at least 2 ways of giving a formal def'n (they turn out to be equiv't).

1) Define what it means for a subset $X \subset \mathbb{R}^k$ to be an n-dim'l mfd., where $n < k$ is any number



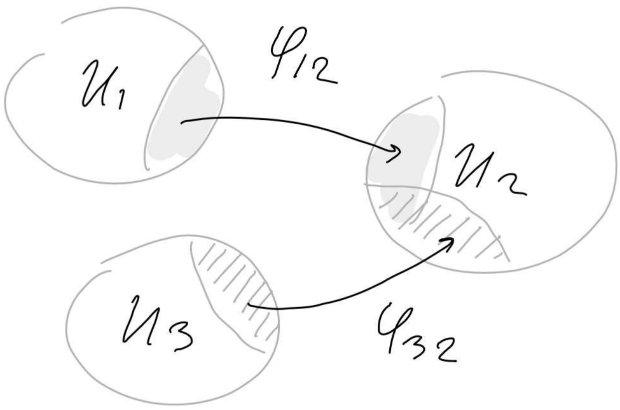
1-dim'l manifold in $\mathbb{R}^2$

1-dim'l mfd in $\mathbb{R}^3$

2-dim'l mfd in $\mathbb{R}^3$

2) Define an n-dim'l mfd as an abstract collection of many copies of open unit k-balls

$$U_i \subset \mathbb{R}^k$$



glued to each other along some subregions (shaded gray in figure) via smooth diffeomorphisms φ_{ij}.

We begin with def. 1), and will work with it for most of the course.
At some point, we'll discuss 2) and why it's equivalent to 1). In fact

$$1) \Rightarrow 2)$$

is quite easy.

Smooth functions & diffeomorphisms

Let $\mathcal{U} \subset \mathbb{R}^k$ be an open set, ie

$$\forall p \in \mathcal{U}, \quad \exists \varepsilon > 0 \quad st \quad B_p(\varepsilon) \subset \mathcal{U}$$

radius ε ball centered at p.

Recall: a function

$$f : \mathcal{U} \longrightarrow \mathbb{R}^n$$

is smooth if it is continuous & all its partial derivatives

$$\frac{\partial f^i}{\partial x^j} \qquad \begin{array}{l} i = 1 \ldots n \\ j = 1 \ldots k \end{array}$$

Here: $f = (f^1, \ldots, f^n)$ components of f.

exist & are continuous on $\mathcal{U}$.

Now let $X \subset \mathbb{R}^k$ be any subset.

<u>Def</u> $f : X \to \mathbb{R}^n$ is smooth if near each point $x \in X$,

f extends to a smooth func. on <u>some</u> open set containing x:

$$\forall x \in X \quad \exists \underset{\text{open}}{U} \subset \mathbb{R}^n, \quad x \in U, \text{ and}$$

$$\exists F : U \to \mathbb{R}^k \text{ smooth s. that } F|_{U \cap X} \equiv f|_X$$

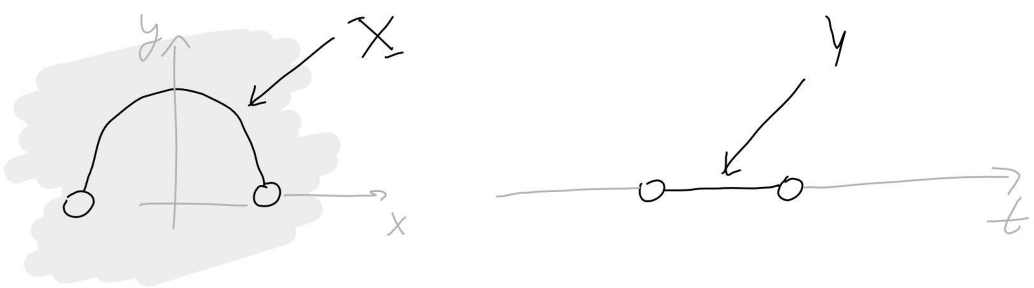
<u>Def</u> Let $X \subset \mathbb{R}^k$, $Y \subset \mathbb{R}^n$ be subsets.
A smooth map $f : X \to Y$ is a <u>diffeomorphism</u> if it is bijective, and $f^{-1} : Y \to X$ is also smooth.

In this case, X and Y are called <u>diffeomorphic</u>

<u>Examples</u> (also see figures in GPo)

1) $X = \{x^2+y^2 = 1, \ y > 0\} \subset \mathbb{R}^2_{x,y}$

$Y = \{t \in (-1, 1)\} \subset \mathbb{R}_t$



X & Y are diffeomorphic.

Indeed, take

$$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto x$$

It is smooth & $f|_X$ is a bijection

with f^{-1} being the following map:

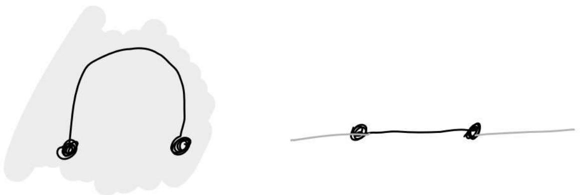
$$t \longmapsto (t, \sqrt{1-t^2})$$

which is also smooth for $t \in (-1, 1)$.

2) In the prev. examples, take the closures

$\bar{X} = \{x^2+y^2 = 1, \ y \geq 0\}$

$\bar{Y} = [-1, 1] \subset \mathbb{R}$



$\bar{X}$ $\bar{Y}$

Then $\bar{X}$ & $\bar{Y}$ are diffeomorphic but the map f from above does not work.

Indeed, $f^{-1} = (t, \sqrt{1-t^2})$ is not differentiable at $t = \pm 1$.

One needs to use a different f.

Before we proceed to the next example, observe the following. Any subset $X \subset \mathbb{R}^k$ is a topological space, where by definition opens subsets of X have the form:

$$\tilde{U} \cap X, \qquad \tilde{U} \subset \mathbb{R}^k \text{ open}$$

<u>Lemma</u> If $f : X \to Y$ is a diffeomorphism then f is also a homeomorphism between X, Y considered as topological spaces.

<u>Proof</u> The def'n of homeomorphism is that f has to be continuous, bijective & f^{-1} is continuous. The lemma follows from the claim below. $\square$

<u>Claim</u>
$$\begin{array}{ccc} \mathbb{R}^k & & \mathbb{R}^n \\ \cup & & \cup \\ X & \xrightarrow{f} & Y \end{array}$$
is smooth. $\Rightarrow f$ is continuous

<u>Proof</u> Continuous means that the reimages of <u>open</u> sets are open. An equivalent definition is that

$\forall y \in Y$, $\forall\, U \subset Y$ open containing y,

$\forall\, x \in X$ s.t. $f(x) = y$ there exists

$V \subset X$ open such that $f(V) \subset U$.

Compare with the usual def'n of a continuous function $f: \mathbb{R} \to \mathbb{R}$:

$\forall \varepsilon \ \exists \delta$ s.t. $f(x-\delta, x+\delta) \subseteq (y-\varepsilon, y+\varepsilon)$.

(It is an exercise to check/recall that this def'n of continuity is equiv. to a std. one)

Now, to prove $f: X \to Y$ is continuous when it's smooth in our sense, start with $y \in Y$, $y \in \mathcal{U} \subset Y$ open, meaning that $\exists \tilde{\mathcal{U}} \subset \mathbb{R}^k$ open s.t. $\tilde{\mathcal{U}} \cap Y = \mathcal{U}$. Let $x \in X$ s.t. $f(x) = y$.

Take $\overset{x}{\overset{\cap}{V}} \subset X$ s.t. $f|_V$ extends to a smooth func:

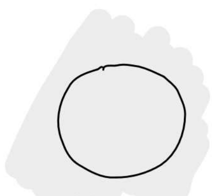
$$F: \underset{\substack{| \\ \text{open in } \mathbb{R}^n,}}{\tilde{V}} \longrightarrow \mathbb{R}^k$$

$$\tilde{V} \cap X = V, \quad F|_X = f.$$

Because F is smooth in the usual sense, it is continuous, in particular

$\exists \tilde{\tilde{V}} \subset \tilde{V}$ open in $\mathbb{R}^n$ s.t. $F(\tilde{\tilde{V}}) \subset \tilde{\mathcal{U}}$. Then $\tilde{\tilde{V}} \cap X$ is open in X and $f(\tilde{\tilde{V}} \cap X) \subset \mathcal{U}$. $\square$

Example

3) $X =$  $\quad Y =$ 

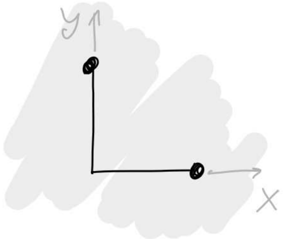
$X \subset \mathbb{R}^2$ circle $\qquad Y \subset \mathbb{R}$ interval.

Then $\nexists$ diffeo $f: X \to Y$.

<u>Proof</u> In fact, there is no homeomorphism $f : X \to Y$. Fix $p \in X$, then f would give a homeomorphism

$$X \setminus \{p\} \longrightarrow Y \setminus \{f(p)\}$$

However, $X \setminus \{p\}$ is <u>connected</u> while $Y \setminus \{f(p)\}$ has 2 connected components, whatever $f(p)$ is. $\square$

4) 

$\underline{X}$

$Y =$ 

interval

There is <u>no</u> diffeomorphism $f : X \to Y$ where

$Y = (-1, 1) \subset \mathbb{R}$.

<u>Proof</u> Denote the inverse $g = f^{-1} : Y \to X$.
Because $X \subset \mathbb{R}^2$, g consists of 2 components:

$$(g_1(t),\ g_2(t)) \in \mathbb{R}^2$$

where $g_1, g_2 : (-1-\varepsilon;\ 1+\varepsilon) \to \mathbb{R}$.
The point $(0,0)$ is in the image of g,

$$\underset{X}{\cap}$$

without loss of generality assume:

$$g_1(0) = 0, \quad g_2(0) = 0.$$

We claim that:

either $g_1'(0) \neq 0$, or $g_2'(0) \neq 0$. $\quad (*)$

Indeed, g has a smooth inverse f, and applying the chain rule to the relation $f \circ g = \mathrm{Id}|_Y$ we get:

$$\left(\partial_x f|_0 \, , \, \partial_y f|_0 \right) \cdot \begin{pmatrix} g_1'(0) \\ g_2'(0) \end{pmatrix} = 1$$

Then $(*)$ follows.

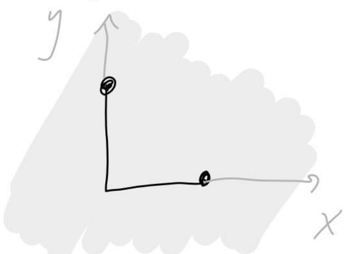
Without loss of generality, assume $g_1'(0) \neq 0$.

We have:

$$g_1(0) = 0$$
$$g_1'(0) \neq 0$$

so the image of g_1 contains $(-\delta, \delta)$ for some δ.

However, look at X:



X is contained in $\{x \geq 0\}$, so we must have $g_1(t) \geq 0$ everywhere.

This is a contradiction. $\square$