1MA259 Differential Topology, Spring 2017
Exercises to Lectures 19-20
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1. Recall the vector field on $\mathbb{R}^2$ with a zero of index 2 at the origin. Draw a perturbation of it which has two zeroes of index 1.

2. Explain why the completeness of the Pontryagin invariant implies the Hopf degree theorem.

3. Consider a planar graph with e edges and v vertices, dividing the plane into f connected components. Show that $v - e + f = 2$.

4. The graph $K_{3,3}$ consists of 6 vertices $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ and 9 edges connecting the vertices a_i with the b_j pairwise. Prove that $K_{3,3}$ is not planar, i.e. cannot be embedded into $\mathbb{R}^2$.
Hint: show that if $K_{3,3} \subset \mathbb{R}^2$, then $f \le 9/2$.

5. Let X, Y be manifolds of the same dimension. Their connected sum $X \sharp Y$ is a manifold obtained as follows: pick some points $p \in X$, $q \in Y$, remove small open neighbourhoods $B(p) \subset X$, $B(q) \subset Y$ of those points, and glue

$$X \setminus B(p) \text{ to } Y \setminus B(q)$$

along their boundary $\partial(X \setminus B(p)) \cong \partial(Y \setminus B(q)) \cong S^{k-1}$. For example, the genus g surface is the connected sum of g copies of T^2. Using the formula for Euler characteristic via triangulations, show that

$$\chi(X \sharp Y) = \chi(X) + \chi(Y) + 2(-1)^{\dim X - 1}.$$

Compute the Euler characteristic of the genus g surface this way.

6. Show that $\chi(SO(n)) = 0$.

7. Let v be a vector field on a manifold X, and $x \in X$ be its zero. Show that dv_x (given by the matrix of partial derivatives of v in some local coordinates) defines a linear map

$$dv_x \colon T_xX \to T_xX$$

which is independent on the choice of local coordinates.

8. We say that a zero x of a vector field v is*non-degenerate* if dv_x is non-degenerate. Show that for a non-degenerate zero,

$$\mathrm{ind}_x v = \text{sign} \det dv_x \in \{\pm 1\}.$$

Hint: show that $(df_t)_x = I - t dv_x$.

9. A subdivision of S^2 into polygons is called *regular* if any vertex of the subdivision is contained in precisely 3 edges, and any two polygons have at most one edge in common. If v, e, f are the numbers of vertices, faces and polygons respectively, prove that

$$v - e + f = 2.$$

Also prove that for a regular subdivision,

$$3p_3 + 2p_4 + p_5 = 12 + \sum_{k \ge 7} (k-6) p_k$$

where p_i is the number of i-gons in the subdivision.

10. A *fullerene* is molecule of carbon, modelled as a regular subdivision of S^2 into pentagons and hexagons. Prove that any fullerene has exactly 12 pentagons.