1MA259 Differential Topology, Spring 2017
Exercises to Lectures 11-12
Dmitry Tonkonog

1. Prove that if $f\colon X \to Y$ is a diffeomorphism between manifolds with boundary, then it maps ∂X diffeomorphically onto ∂Y.

2. Let X^k be a manifold with boundary, $x \in \partial X$ a point, $V \subset H^k$ a neighbourhood of the origin in the upper half-space, and

$$\phi\colon V \to X^k$$

be a parametrisation of X near x, $\phi(0) = x$. (It is a diffeomorphism onto its image, which is a neighbourhood of x in X.) Denote by $H_x(X) \subset T_xX$ the half-space in T_xX obtained as the image of $H^k \subset \mathbb{R}^k$ under the linear map $d\phi_0\colon \mathbb{R}^k \to T_xX$:

$$H_x(X) = d\phi_0(H^k).$$

Prove that $H_x(X) \subset T_xX$ does not depend on the choice of the parametrisation ϕ.

3. Let $X \subset \mathbb{R}^n$ be a manifold with boundary. Show that there are precisely two unit vectors in $T_xX \subset \mathbb{R}^n$ that are orthogonal to $T_x\partial X$, and one can pick one of those two vectors canonically by the requirement that it lies in $H_x(X)$. This vector is called the *inward normal vector* to the boundary. Prove that it depends smoothly on the point of the boundary.

If you know what an orientation on a manifold is, prove the following: if X is orientable, then ∂X is orientable. (If you don't, this part of the exercise will probably reappear later.)

4. Construct a diffeomorphism from the closed unit disk to itself with a unique fixed point, and such that this fixed point lies on the boundary.

5. Prove that there are two different points on Earth where the temperature and the height above/below sea level are the same.

6. Show that any map $f\colon S^n \to S^n$ with degree 1 (modulo 2) must carry some pair of antipodal points to a pair of antipodal points. By a pair of antipodal points, one means a pair of form $(x, -x)$ where $x \in \mathbb{R}^{n+1}$.

Hint: show that otherwise, f is homotopic to a map that takes every pair of antipodal points to the same point.

7. Is every map $f\colon S^1 \times S^1 \to S^2$ null-homotopic?

The exercises below are marked with a star because they require some material beyond the scope of this course.

8*. Prove that any map $f\colon S^2 \to S^1 \times S^1$ is null-homotopic.

Hint: use the universal covering $\mathbb{R}^2 \to S^1 \times S^1$, and the theorem about lifting maps to covering spaces. If you have not met these facts before, you may consider this an an opportunity to learn about them.

9*. Prove that the degree of any map $f\colon S^3 \to S^2 \times S^1$ is zero.

Hint: use the universal covering $\mathbb{R}^2 \to S^1 \times S^1$; same remark as above applies.