1MA259 Differential Topology, Spring 2017
Exercises to Lectures 08-09
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1. We discussed earlier how to glue the torus T^2 from the square. Prove there is a Morse function on T^2 with one minimum, one maximum, and two index 1 critical points such that both index 1 critical points are on the same level set. Define such a function f by a picture showing sufficiently many levels sets of f inside the square, paying attention to the condition that the level sets must correctly match across the boundary of the square.

2. The Klein bottle is a 2-dimensional manifold which can be obtained by gluing the opposite pairs of sides of the square (like for the torus), but one pair of sides is glued "with a twist". If you don't have a picture in mind, look it up or ask someone. Draw a picture showing the square with (sufficiently many) level sets of some function on the Klein bottle which has one minimum, one maximum, and two index 1 critical points. Compute the Betti numbers of the Klein bottle.

3. In the lectures I sketched how to define the homology $H^i(X)$ with coefficients in $\mathbb{Z}/2$, and mentioned that one can perform the same construction over $\mathbb{Z}$ by carefully putting in the correct signs in all definitions (which I didn't explain). The homology of the Klein bottle K over $\mathbb{Z}$ is known to be: $H^0(K) = 0$, $H^1(K) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $H^2(K) = \mathbb{Z}/2\mathbb{Z}$. Knowing these homologies, can you guess how the differential of the Morse complex for the Klein bottle looks like (using the Morse function from the previous example), by counting each flowline with sign $+1$ or -1 in a way which gives this answer?

4. Prove that for a compact manifold X of dimension n, $h^i(X) = h^{n-1}(X)$. This is a simple version of *Poincaré duality.*

5. Does there exist a Morse function on S^2 with one minumum, 2 maxima and:

- no other critical points?
- one index 1 critical point?
- twelve index 1 critical points?

6. Prove that there is a smooth function on the torus which has one non-degenerate minumum and maximum, and only one additional critical point (possibly degenerate). Can this third critical point be non-degenerate?

7. Suppose $f_t \colon X \to \mathbb{R}$ is a family of functions smoothly depending on the parameter $t \in (-1, 1)$ and such that f_0 is Morse. Show that f_ϵ is also Morse, for any sufficiently small ϵ.

8. Modifying the proof of Whitney embedding theorem, show that every n-dimensional manifold can be immersed in $\mathbb{R}^{2n}$.