1MA259 Differential Topology, Spring 2017
Exercises to Lectures 06-07
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1. Define $f\colon \mathbb{R}^n \to \mathbb{R}$ by the formula:

$$\begin{cases} e^{-1/(1-|x|^2)}, & |x| < 1, \\ 0, & |x| \geq 1. \end{cases}$$

Here $x \in \mathbb{R}^n$ and $|x|$ is the norm. Check that f is smooth.

2. Let $K \subset \mathbb{R}^n$ be a compact set. Prove that there is a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $f^{-1}(0) = K$. Deduce that there is an n-dimensional manifold $X \subset \mathbb{R}^{n+1}$ such that $X \cap (\mathbb{R}^n \times \{0\}) = K$.

3. Let $X, Z \subset Y$ be transverse submanifolds, and $x \in X \cap Z$. Prove that $T_x(X \cap Z) = T_x X \cap T_x Z$.

4. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be smooth maps, and g is transversal to a submanifold $W \subset Z$. Show that f is transversal to $g^{-1}(W)$ if and only if $g \circ f$ is transversal to W.

5. Let $f\colon X \to X$ be a smooth map, and let $\dim X = n$. Consider the following n-dimensional submanifolds of $X \times X$, called the *diagonal* and the *graph of f* respectively:

$$\Delta = \{(x, x) \in X \times X : x \in X\},$$
$$\Gamma_f = \{(x, f(x)) \in X \times X : x \in X\}.$$

Verify that the intersections $\Delta \cap \Gamma_f$ are in bijection with the *fixed points* of f, i.e. points $x \in X$ such that $f(x) = x$.

We say that $f\colon X \to X$ is a *Lefschetz map* if for any fixed point $x \in X$ of f, the differential $df_x\colon T_x X \to T_x X$ does not have $+1$ among its eigenvalues. Prove that Δ intersects Γ_f transversally if and only if f is a Leschetz map.

6. Suppose $f\colon X \to X$ is a Lefschetz map, and X is compact. Prove that f has finitely many fixed points.

7. Prove that the complement to a measure zero set is dense.

8. Prove that the set of rational numbers $\mathbb{Q} \subset \mathbb{R}$ has measure 0.

9. Analyse the critical behaviour of the following functions at the origin. Is the critical point nondegenerate? Is it isolated? Is it a local maximum or minimum?

$$f(x, y) = x^2 + 4y^3,$$
$$f(x, y) = x^2 - 2xy + y^2,$$
$$f(x, y) = x^2 + y^4,$$
$$f(x, y) = x^2 + 11xy + y^2/2 + x^6,$$
$$f(x, y) = 10xy + y^2 + 75y^3.$$

10*. Let X be a compact manifold. Prove that there exist Morse functions on X which take distinct values at distinct critical points. *Hint:* start with some Morse function, and let $x_i \in X$ be its critical points. Let $\rho_i\colon X \to \mathbb{R}$ be smooth functions supported in some neighbourhoods U_i of x_i, such that $U_i \cap U_j = \emptyset$. (Here, f_i *is supported in* U_i means that $f_i \equiv 0$ away from U_i. Such functions can be constructed by passing to a chart and using Problem 1.) Prove that $g = f + \sum a_i \rho_i$ has distinct critical values, where a_i are sufficiently small generically chosen constants.