1MA259 Differential Topology, Spring 2017
Exercises to Lectures 01, 02
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1. In the lectures we proved that the set

$$X = \{0 \le x \le 1,\ y = 0\} \cup \{x = 0,\ 0 \le y \le 1\} \subset \mathbb{R}^2(x, y)$$

is not diffeomorphic to $Y = [-1, 1] \subset \mathbb{R}$. Prove the same statement for the set

$$X = \{0 \le x \le 1,\ y = 0\} \cup \{x = -y,\ 0 \le y \le 1\} \subset \mathbb{R}^2(x, y).$$

2. Let

$$X = \{(a_1, \ldots, a_n, 0, \ldots 0)\} \subset \mathbb{R}^k$$

be the linear subspace $\mathbb{R}^n \subset \mathbb{R}^k$. Prove that a smooth function $f\colon X \to \mathbb{R}^m$ in the sense defined in the lectures is the same as a smooth function $\mathbb{R}^n \to \mathbb{R}^m$ in the usual sense.

3. Consider subsets $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$, $Z \subset \mathbb{R}^k$, and let $f\colon X \to Y$, $g\colon Y \to \mathbb{Z}$ be smooth maps. Then the composition $g \circ f\colon X \to Z$ is smooth. If f, g are diffeomorphisms, so is $g \circ f$.

4. Let

$$B(a) = \{x_1^2 + \ldots + x_k^2 < a\} \subset \mathbb{R}^k$$

be the open ball of radius a. Check that the map

$$f(x) = \frac{ax}{\sqrt{a^2 - |x|^2}}, \quad x \in \mathbb{R}^k,$$

where $|x|^2 = x_1^2 + \ldots + x_k^2$, defines a diffeomorphism $f\colon B(a) \to \mathbb{R}^k$ onto the whole of $\mathbb{R}^k$.

5. Check that the map $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$, is not a diffeomorphism, although f is smooth and bijective.

6. Prove that the union of two complementary n-dimensional subspaces in $\mathbb{R}^{2n}$,

$$X = (\mathbb{R}^n \times \{0\}) \cup (\{0\} \times \mathbb{R}^n) \subset \mathbb{R}^{2n},$$

is not a manifold. (Hint: what happens if you remove the origin?)

8. Prove that the unit sphere in $S^n \subset \mathbb{R}^{n+1}$ is not diffeomorphic to $\mathbb{R}^n$. (Hint: $\mathbb{R}^n$ is not compact.)

9. Prove that the map $f = (\sin u, \cos u)$, $u \in (0, 2\pi)$, is a chart for $S^1 \setminus \{1\} \subset \mathbb{R}^2$, where S^1 is the unit circle. Define an analogous chart which would cover $\{1\} \in S^1$.

10. Let $S^1_r \subset \mathbb{R}^3$ be the circle of radius $r > 1$ lying in the horizontal plane $\mathbb{R}^2 \subset \mathbb{R}^3$, and let $X \subset \mathbb{R}^3$ be the set consisting of points at distance 1 from S^1_r. Prove that X is a manifold diffeomorphic to $S^1 \times S^1$. Hint: show that X coincides with the set

$$\{(r + \sin u)\cos v,\ (r + \sin u)\sin v,\ \cos u) : u, v \in [0, 2\pi]\}.$$

11. Consider the set $SO(2; \mathbb{R})$ of 2×2 orthogonal matrices as a subset of $\mathbb{R}^4$, by taking the matrix coefficients. Write the down the equations defining this subset, $SO(2; \mathbb{R}) \subset \mathbb{R}^4$. Prove that $SO(2; \mathbb{R})$ is a manifold and is diffeomorphic to S^1.

12. Recall the stereographic projection $S^n \setminus \{\textit{top point}\} \to \mathbb{R}^n$ mentioned in the lectures. Write a formula for it, and check that it is a diffeomorphism. See Exercise 12 from [Guillemin-Pollack, Lecture 1] or the lecture notes for a picture.

13*. The product $S^2 \times S^2$ is a manifold which lies in $\mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6$. Prove that there is a manifold $X \subset \mathbb{R}^5$ which is diffeomorphic to $S^2 \times S^2$.