1MA259 Differential Manifolds, Spring 2017. Lecturer: Dmitry Tonkonog

Homework Assignment 2

Submit by: **29th March, 01:00pm**

Problem 1, 2 points. Let M be a manifold of dimension m, and S^p be the p-dimensional sphere. If $m < p$, show that every smooth map $f\colon M \to S^p$ is null-homotopic.

Hint: use Sard's theorem to argue that the image of f avoids some point.

Problem 2, 2 points. Exhibit a map $S^1 \to U(n)$ which is not null-homotopic. Here $U(n)$ is the manifold consisting of complex unitary $n \times n$ matrices.

Hint: use the determinant map $U(n) \to S^1 = \{z \in \mathbb{C} : |z| = 1\}$ to prove that the map you have constructed is not null-homotopic.

Problem 3, 2 points. Prove that there is a complex number z such that

$$z^7 + \cos(|z|^2)(1 + 93z^4) = 0$$

Hint: prove that the restriction of the map to some circle has non-zero winding number around the origin.

Problem 4, 2 points. Consider the map $f\colon S^1 \to SO(3)$ given by:

$$t \mapsto \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad t \in [0, 2\pi].$$

Prove that this map is not null-homotopic by showing that there exists a 2-dimensional manifold X and a smooth map $g\colon X \to SO(3)$ such that $I_2(f, g) = 1$.

Hint: try composing rotations around the x-axis and the y-axis.

Remark: the map $f(2t)$, $t \in [0, 2\pi]$, which goes "twice around" the circle in the image of f, is null-homotopic. This is sometimes referred to as the waiter's trick; for a demonstration, you are invited to watch a beautiful youtube movie "Air on the Dirac Strings".

Problem 5, 4 points. Let D^{n+1} be the closed unit ball of dimension $n + 1$, S^n the unit n-sphere, and Y a manifold. Carefully prove that a map $f\colon S^n \to Y$ extends to a smooth map $D^{n+1} \to Y$ if and only if f is null-homotopic.

Hint: you can use, without proof, the fact that if one removes an arbitrarily small closed neighbourhood of an interior point $p \in D^{n+1}$, one gets manifold a diffeomorphic to $S^n \times [0, 1]$. Second hint: show that by applying a homotopy, one can make any map between two manifolds constant in a neighbourhood of a given point.

Problem 6, 4 points. Given disjoint compact manifolds without boundary M^m, $N^n \subset \mathbb{R}^{k+1}$, the linking map is defined by

$$\lambda\colon M \times N \to S^k, \quad \lambda(x, y) = \frac{x - y}{\|x - y\|}, \ x \in M, \ y \in N$$

where the subtraction and the norm are applied to vectors in $\mathbb{R}^{k+1}$. From now on, assume that $m + n = k$. The (modulo 2) degree of λ is called the (modulo 2) linking number, and is denoted by $l(M, N) \in \{0, 1\}$.

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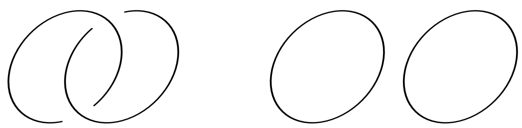
(a). Prove that $l(M, N) = l(N, M)$.

(b). Suppose f_0, f_1 are *isotopic* embeddings of the disjoint union $M \sqcup N \to \mathbb{R}^{k+1}$. Prove that $l(f_0(M), f_0(N)) = l(f_1(M), f_1(N))$.

Recall that an isotopy from f_0 to f_1 is a homotopy $f_t\colon M \sqcup N \to \mathbb{R}^{k+1}$, $t \in [0,1]$, from f_0 to f_1 such that each f_t is an embedding. In particular, each f_t is injective on $M \sqcup N$.

(c). Let $f\colon S^1 \sqcup S^1 \to \mathbb{R}^3$ be the embedding whose image is shown in Figure (a), and $g\colon S^1 \sqcup S^1 \to \mathbb{R}^3$ be the embedding whose image is shown in Figure (b). Prove that f and g are not isotopic by computing the linking numbers. An explanation with reference to the picture is sufficient.

(a) (b)



Problem 7, 4 points. Let $Y \subset \mathbb{R}^n$ be a manifold, $X \subset Y$ a submanifold and $x \in X$. We define the normal space $N_x X \subset T_x Y$ of X (considered as a submanifold of Y) to be the linear orthogonal complement of the subspace $T_x X \subset T_x Y$ taken inside $T_y Y$. Now define

$$N_Y X = \{(x, v) : x \in X,\ v \in N_x X\} \subset \mathbb{R}^n \times \mathbb{R}^n.$$

We call this space the *normal bundle to X in Y*.

(a). Prove that $N_Y X$ is a manifold.

Hint: explain how to modify or extend the proof, found in the lectures, that the normal bundle of a manifold in $\mathbb{R}^n$ is itself a manifold.

(b). Let $\Delta \subset Y \times Y$ be the diagonal, consisting of points (y, y), $y \in Y$. Prove that $N_{Y \times Y}\Delta$ is diffeomorphic to $T\Delta$, where $T\Delta$ is the tangent bundle of Δ.

Submitting your work. You can submit your work in one of the two ways. First, you can submit handwritten solutions to my postbox (labelled: "Dmitry Tonkonog") in the Department of Mathematics, Floor 4 of Angstrom Laboratory building. If my postbox is full, please put your work on the common table inside the post room. Second, you can prepare a pdf file with solutions, typeset using latex, and send it to dmitry.tonkonog@math.uu.se. Both ways, the solutions must be submitted by the deadline specified above. Do not forget to write your name on your work.