1MA259 Differential Topology, Spring 2017

Exercises to Lectures 03, 04

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1. If $f\colon X \to Y$ is a diffeomorphism between two manifolds, prove that $d_x f\colon T_x X \to T_{f(x)} Y$ is an isomorphism between vector spaces for each $x \in X$.

2. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere. For each point $x \in X$, compute $T_x S^2$.

3. Compute the tangent plane to the paraboloid in $\mathbb{R}^3$ given by

$$x^2 + y^2 - z^2 = a,$$

where $a > 0$ is fixed, at the point $(\sqrt{a}, 0, 0)$.

4. Prove that any submersion $f\colon X \to Y$ is an *open map.* That is, for any open set $U \subset X$, $f(U)$ is open in Y.

5. Prove that if $f\colon X \to Y$ is a submersion, X is compact and Y is connected, then f is surjective.

6. Prove that $0 \in \mathbb{R}^3$ is the only point where the derivative of the map $f\colon \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x, y, z) = x^2 + y^2 - z^2$$

vanishes. Prove that if a, b are non-zero and have the same sign, $f^{-1}(a)$ is diffeomorphic to $f^{-1}(b)$. (Hint: consider the scalar multiplication by $\sqrt{b/a}$ in $\mathbb{R}^3$.) Draw a picture of the level sets $f^{-1}(a)$ for $a < 0$, $a = 0$, $a > 0$.

7. Prove that the set of all 2×2 matrices of rank one is a 3-dimensional manifold in $\mathbb{R}^4$. Recall that $\mathbb{R}^4$ is canonically identified with the set of all 2×2 matrices. (Hint: prove that the determinant function is a submersion away from the point representing the zero matrix.)

8. Suppose Z is an l-dimensional submanifold of a k-dimensional manifold X, and $z \in Z$. Show that there exists a local parametrisation $\{x_1, \ldots, x_k\}$ for a neigbourhood V of z in X such that $Z \cap U$ is defined by the equations:

$$x_{l+1} = \ldots = x_k = 0.$$

Note: I promised an exercise to show that $SO(n)$ is a manifold, but I decided to explain it on Lecture 4, as it is quite important.