1MA259 Differential Topology, Spring 2017. Lecturer: Dmitry Tonkonog

Homework Assignment 3

Submit by: **4th May, 01:00pm**

Problem 1, 3 points.

(a). Let X^n be a manifold such that TX is trivial: that is, $TX \cong X \times \mathbb{R}^n$. Prove that $\chi(X) = 0$. Conclude that TS^2 is non-trivial.

(b). Show that $TS^2 \times \mathbb{R}$ is diffeomophic to $S^2 \times \mathbb{R}^3$.

Problem 2, 3 points. Let X be a manifold. We define the *unit tangent bundle* of X as the submanifold in the tangent bundle

$$TX = \{(x, v) : x \in X,\ v \in T_x X\}$$

given by the equation $\|v\| = 1$, where $\|\cdot\|$ is the Euclidean norm. Prove that the unit tangent bundle of S^2 is diffeomorphic to $SO(3)$.

Note: please include an explanation why the maps you construct are smooth and not just continuous.

Problem 3, 3 points. Consider smooth maps between closed oriented manifolds of the same dimension: $X \xrightarrow{f} Y \xrightarrow{g} Z$. Prove that

$$\deg(g \circ f) = \deg f \cdot \deg g \in \mathbb{Z}.$$

Problem 4, 5 points. Consider disjoint closed manifolds M^m, $N^n \subset \mathbb{R}^{k+1}$; assume that $m + n = k$ and M, N are oriented. Let

$$\lambda\colon\ M \times N \to S^k$$

be the linking map from Homework Assignment 2. Assume that an orientation on S^k has been fixed, we define the (integral-valued) linking number:

$$l(M, N) = \deg \lambda \in \mathbb{Z}.$$

Note: the difference from Homework Assignment 2 is that now $l(M, N) \in \mathbb{Z}$ rather than $l(M, N) \in \{0, 1\}$.

As a slight modification of this definition, suppose M, N are submanifolds of the unit sphere S^{k+1}. Again, assume that $m + n = k$, and M, N are oriented. One defines the linking number $l(M, N) \in \mathbb{Z}$ as follows. Pick a point $p \in S^{k+1}$ away from M and N. Consider any diffeomorphism (for example, the stereographic projection)

$$h\colon\ S^{k+1} \setminus \{p\} \to \mathbb{R}^{k+1}.$$

Then put

$$l(M, N) = l(h(M), h(N)) \in \mathbb{Z},$$

where the linking number in the right hand side has been defined above. Let us take it for granted that $l(M, N)$ does not depend on the choice of p and the diffeomorphism h.

Now let $f\colon\ S^{2p-1} \to S^p$ be a smooth map, and $y, z \in S^p$ be regular values for f. By the preimage theorem for submersions, $f^{-1}(y)$, $f^{-1}(z)$ are $(p-1)$-dimensional submanifolds of S^{2p-1}. Therefore the linking number $l(f^{-1}(y), f^{-1}(z)) \in \mathbb{Z}$ is defined.

PROCEED TO THE SECOND PAGE

(a). Prove that for any other regular value y' sufficiently close to y,

$$l(f^{-1}(y'), f^{-1}(z)) = l(f^{-1}(y), f^{-1}(z)).$$

Hint: the results from Lecture 20 will be helpful.

(b). Let $g\colon S^{2p-1} \to S^p$ be another smooth map. Suppose $y, z \in S^p$ are regular values for both f and g, and that for all $x \in S^p$ we have:

$$\|f(x) - g(x)\| < \|y - z\|,$$

where the norm is computed in $\mathbb{R}^{p+1} \supset S^p$. Prove that

$$l(f^{-1}(y), f^{-1}(z)) = l(g^{-1}(y), f^{-1}(z)) = l(g^{-1}(y), g^{-1}(z)).$$

(c). Prove that $l(f^{-1}(y), f^{-1}(z))$ does not depend on the choice of y, z (as long as they are regular values), and does not change if we replace f by a homotopic map.

The invariant $l(f^{-1}(y), f^{-1}(z))$ thus defined is called *the Hopf invariant* $h(f) \in \mathbb{Z}$.

Problem 5, 3 points. The Hopf fibration $\pi\colon S^3 \to S^2$ is defined by

$$\pi(x_1, x_2, x_3, x_4) = h^{-1}\left(\frac{x_1 + ix_2}{x_3 + ix_4}\right)$$

where $h\colon S^2 \setminus \{p\} \to \mathbb{C}$ is the stereographic projection, and $S^3 = \{x_1^2 + \ldots + x_4^2\} \subset \mathbb{R}^4$. Prove that

$$h(\pi) = 1,$$

where h is the Hopf invariant from Problem 4.

Problem 6, 4 points. Consider smooth maps $S^{2p-1} \xrightarrow{f} S^p \xrightarrow{g} S^p$. Prove that

$$h(g \circ f) = h(f) \cdot (\deg g)^2,$$

where h is the Hopf invariant from Problem 4.

Problem 7, 5 points. Let X be a manifold without boundary, and $f\colon X \to X$ a smooth map such that $f^2 = f \circ f$ is the identity. Such a map is called an *involution.* Let $M \subset X$ be the *fixed point set* of f:

$$M = \{x \in X\colon\ f(x) = x\}.$$

Prove that each connected component of M is a submanifold of X.

Note: different connected components of M may have different dimensions.

Hint: assume $X = \mathbb{R}^n$ and $0 \in \mathbb{R}^n$ is a fixed point of f. Let $df_0\colon \mathbb{R}^n \to \mathbb{R}^n$ be the differential at the origin, and define $g\colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$g(x) = x + (df_0)(f^{-1}(x)).$$

Show that g is a local diffeomorphism near the origin; and that $g(f(x)) = (df_0)(g(x))$ — so f becomes linear after the coordinate change given by g.

Problem 8, 4 points. Is there an involution on the 2-sphere with exactly 4 distinct fixed points?

Hint: using your solution to the previous problem or otherwise, show that such an involution is a Lefschetz map. What can you say about the signs of its fixed points?

Submitting your work. You can submit handwritten solutions to my postbox (labelled: "Dmitry Tonkonog") in the Department of Mathematics, Floor 4 of Angstrom Laboratory building, or send your solutions in pdf format to dmitry.tonkonog@math.uu.se.