1MA259 Differential Manifolds, Spring 2017. Lecturer: Dmitry Tonkonog

Homework Assignment 1

Submit by: **24th February, 01:00pm**

Problem 1, 3 points. Let $f\colon X \to Y$ be a submersion between manifolds of the same dimension, and assume that X is compact. Prove that for any $y \in Y$, the full preimage $f^{-1}(y)$ consists of a finite set of points.

Problem 2, 3 points. Let $X \subset \mathbb{R}^3$ be a compact 2-dimensional manifold. Prove that there exist at least two distinct points $x_1, x_2 \in X$ such that the tangent plane $T_{x_i}X$ is spanned by the vectors $(1, 0, 0)$ and $(0, 1, 0)$, for each $i = 1, 2$.

Hint: consider the height function $f(x, y, z) = z$. Explain why the maximum and the minimum must be critical points.

Problem 3, 3 points. Can the equation $x^2+y^2+z^2 = \sin(\frac{\pi}{2}x)\cos(\frac{\pi}{2}y)$ be solved uniquely for x in terms of y, z near the point $x = 1$, $y = 0$, $z = 0$?

Hint: use the inverse function theorem. Explain which map you apply this theorem to.

Problem 4, 4 points. Let $f\colon X \to \mathbb{R}^n$, $g\colon Y \to \mathbb{R}^n$ be two embeddings, $\dim X+\dim Y > n$. Prove that there exists a vector $v \in \mathbb{R}^n$ such that the manifolds $f(X)$ and $g(Y) + v$ intersect transversally. Here $g(Y) + v$ is obtained by from $g(Y)$ by parallel translation along v. Explain that moreover, v can be chosen to have arbitrarily small norm.

Hint: consider the map $f + g\colon X \times Y \to \mathbb{R}^n$, and apply Sard's theorem.

Problem 5, 4 points. Let $M(n) = \mathbb{R}^{n^2}$ be the space of all $n \times n$ matrices, and $F\colon M(n) \to \mathbb{R}^n$ be the map sending a matrix to its first column, considered as a vector in $\mathbb{R}^n$. Clearly, F restricts to a smooth map $f\colon O(n) \to S^{n-1}$, where $O(n) \subset M(n)$ is the manifold consisting of orthogonal matrices, and $S^{n-1} \subset \mathbb{R}^n$ is the unit sphere.

(a). Prove that f is a submersion at the point $I \in O(n)$, where I is the unit matrix.

Hint: use the fact that the tangent space $T_I O(n)$ consists of skew-symmetric matrices, as shown in Lecture 4.

(b). Prove that f is a submersion at all points of $O(n)$.

Hint: use the action of $O(n)$ on itself by left multiplication to reduce the problem to the previous case. If you use this trick, explain and justify it.

Problem 6, 3 points. Let $M(n) = \mathbb{R}^{n^2}$ be the space of all $n \times n$ matrices. Prove that 1 is a regular value for the map computing the determinant of a matrix, $\det\colon M(n) \to \mathbb{R}$. Conclude that the space $SL(n)$ consisting of matrices with determinant 1 is a manifold.

Submitting your work. You can submit your work in one of the two ways. First, you can submit handwritten solutions to my postbox (labelled: "Dmitry Tonkonog") in the Department of Mathematics, Floor 4 of Angstrom Laboratory building. If my postbox is full, please put your work on the common table inside the post room. Second, you can prepare a pdf file with solutions, typeset using latex, and send it to dmitry.tonkonog@math.uu.se. Both ways, the solutions must be submitted by the deadline specified above. Do not forget to write your name on your work.