Bank of True - False Questions

Sections 1.1-1.9.

1) **T-F** Every transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is linear.

2) **T-F** A linear transformation maps the origin to itself. In other words, if T is linear then $T(0) = 0$.

3) **T-F** The composition of two linear transformations is linear. In other words if T_1 and T_2 are linear transformations then $T_1 \circ T_2$ is a linear transformation where $T_1 \circ T_2(x) = T_1(T_2(x))$.

4) **T-F** If A is a 3×2 matrix then the transformation $x \mapsto Ax$ is one to one.

5) **T-F** If A is a 3×2 matrix then the transformation $x \mapsto Ax$ is onto.

6) **T-F** The columns of the standard matrix for a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ are the images of the columns of the $n \times n$ identity matrix.

7) **T-F** Consider a linear transformation T. Then T is one to one if and only if the column vectors of its standard matrix are linearly independent. (They are linearly independent if and only if the span of the column vectors of A is n dimensional.)

8) **T-F** Consider a linear transformation T such that its standard matrix A is an $m \times n$ matrix. (This means $T : \mathbb{R}^n \to \mathbb{R}^m$). T is onto if and only if the span of the column vectors of A is m dimensional.

   1) **False.** Consider $T(x) : \mathbb{R} \to \mathbb{R}$ where $T(x) = x^2$.

   2) **True.**
   $$T(0) = T(x - x) = T(x) - T(x) = 0$$

   3) **True.** Let's say the standard matrix of T_1 is A_1 and the standard matrix of T_2 is A_2 then the composition $T_1 \circ T_2$ is given by left multiplication by $A_1 A_2$ because $T_1 \circ T_2(x) = T_1(T_2(x)) = T_1(A_2(x)) = A_1 A_2 x$. Transformations given by matrix multiplication are linear by definition of matrix multiplication.

   4) **False.** It can be one to one. For instance if $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ then it is, but it

doesn't have to be. If $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ -7.3 & 0 \end{bmatrix}$ then it is not because if $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $x' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ then $Ax = Ax' = \begin{bmatrix} 1 \\ 2 \\ -7.3 \end{bmatrix}$.

5) **False.** It cannot be onto because the domain is $\mathbb{R}^2$ which is 2 dimensional so the image is at most 2 dimensional. However the target set $\mathbb{R}^3$ is three dimensional.

6) **True.** See Theorem 10 pg. 76.

7) **True.** We can write the standard matrix as $A = [c^1, \ldots, c^n]$ where c^i is the i th column. Then T is one to one if and only if $Av = Av'$ implies $v = v'$. But $Av = Av'$ is equivalent to $A(v - v') = 0$ (here the right hand side is the 0 vector of length m). We can write $v - v' = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ then $A(v - v') = 0$ is equivalent to $a_1 \cdot c^1 + \cdots + a_n \cdot c^n = 0$ and $v = v'$ is equivalent to saying $a_1 = \cdots = a_n = 0$. So saying T is one to one is equivalent to saying if $a_1 \cdot c^1 + \cdots + a_n \cdot c^n = 0$ then $a_1 = \cdots = a_n = 0$ which by definition means $\{c_1, \ldots, c_n\}$ are linearly independent.

8) **True.** The image of T is the span of the columns of A so T is onto if and only if the span of the columns of A is equal to $\mathbb{R}^m$.

Sections 2.1-2.3.

1) **T-F** If A and B are $n \times n$ matrices, then $AB = BA$.

2) **T-F** Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set of vectors in $\mathbb{R}^6$. Then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is also a linearly dependent set.

3) **T-F** If A is a 2×3 matrix then A is never onto.

4) **T-F** If A is an $n \times n$ square matrix and $A^\top A = 2I$, then A is invertible.

5) **T-F** If A and B are invertible $n \times n$ square matrices then $(A + B)^{-1} = A^{-1} + B^{-1}$.

1) **False.** For example, try $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

2) **False.** For example, take $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

3) **False.** For example, the 2×3 matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is onto.

4) **True.** The inverse of A is $\frac{1}{2}A^\top$. To verify this, we can observe that $\frac{1}{2}A^\top A = \frac{1}{2}(2I) = I$ and (to verify that it is also a right inverse, although this is not necessary for square matrices) we can compute $A(\frac{1}{2}A^\top) = \frac{1}{2}(AA^\top) = \frac{1}{2}(A^\top A)^\top = \frac{1}{2}(2I)^\top = I$.

5) **False.** In fact, $A+B$ may not even be invertible! Try $A = I$ and $B = -I$

Sections 3.1-3.3.

1) **T-F** If A has determinant $\det(A) = 0$, then two rows of A are multiples of each other.
2) **T-F** If A is obtained by multiplying a row of B by 5, then $\det(A) = 5\det(B)$.
3) **T-F** If A is obtained by subtracting column 2 from column 1 in B, then $\det(A) = -\det(B)$.
4) **T-F** If A^T has nonpositive determinant $\det(A^T) \leq 0$, then $\det(A^T A) \leq 0$.
5) **T-F** If A is a 3×3 matrix obtained by switching rows 2 and 3 of the identity matrix I, then A has determinant $\det(A) = 1$.
6) **T-F** If the rows of A are linearly dependent, then A has $\det(A) = 0$.
7) **T-F** If the columns of A are linearly independent, then A has $\det(A) \neq 0$.
8) **T-F** If A has $\det(A) = a$, and B has $\det(B) = b$, then $\det(A + B) = a + b$.
9) **T-F** If A is a 4×4 matrix, then $\det(-A) = \det(A)$.
10) **T-F** If A is an $n \times n$ matrix and the columns of A sum to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, then A is invertible.
11) **T-F** If A and B are $n \times n$ matrices, $\det(A + B^T) = \det(A + B)$.
12) **T-F** If the linear transformation $\mathbb{R}^n \to \mathbb{R}^n$ defined by A is injective (one-to-one), then A is surjective (onto).
13) **T-F** If $\det(A) = 1$ and A' is the adjugate matrix of A, then $AA' = I$, where I is the identity matrix.

1) **False**. Not necessarily. Take for example $\begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 4 \\ 2 & 2 & 4 \end{bmatrix}$
2) **True**. If one row of A is multiplied by a scalar k to produce B, then $\det(B) = k\det(A)$.
3) **False**. If a multiple of one row of A is added to another row to produce a matrix B, then $\det(B) = \det(A)$.

4) **False**. If $\det(A^T) \leq 0$, then $\det(A) \leq 0$, so $\det(A^T A) = \det(A^T)\det(A) \geq 0$.

5) **False.** If two rows of I are interchanged to produce A, then $\det(A) = -\det(I) = -1$.

6) **True**. The matrix A is not invertible, and so has $\det(A) = 0$.

7) **True**. The matrix A is invertible, so has $\det(A) \neq 0$.

8) **False**. The determinant of a matrix is not distributive over addition.

9) **True**: $\det(-A) = (-1)^4 \det(A)$.

10) **False:** Not necessarily, take $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

11) **False:** not necessarily.

12) If the linear transformation $\mathbb{R}^n \to \mathbb{R}^n$ defined by A is injective (one-to-one), then A is surjective (onto).

13) If $\det(A) = 1$ and A' is the adjugate matrix of A, then $AA' = I$, where I is the identity matrix.

Sections 4.1-4.6.

1) **T-F** The null space of an $m \times n$ matrix is in $\mathbb{R}^m$.
2) **T-F** The column space of an $m \times n$ matrix is in $\mathbb{R}^m$.
3) **T-F** If the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for some nonzero $\mathbf{b}$, then $\text{Col}(A) = \{\mathbf{0}\}$.
4) **T-F** Let $T : \mathbb{R}^3 \to R^3$ be a linear transformation. If $\ker(T) = \{0\}$, then $\text{range}(T) = \mathbb{R}^3$.
5) **T-F** Let A be an $n \times n$ matrix, and suppose that $\text{Col}(A) = \text{Null}(A)$. Then $\text{Null}(A^2) = \mathbb{R}^n$.
6) **T-F** The set $\mathcal{S} = \{1, x\}$ is a basis for the vector space of polynomials of degree ≤ 2, P_2.
7) **T-F** For $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is in $\text{null}(A)$.
8) **T-F** For $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is in $\text{col}(A)$.
9) **T-F** The set of invertible $n \times n$ matrices is a vector subspace of the set of all $n \times n$ matrices, $\text{Mat}_n(\mathbb{R})$.
10) **T-F** Let T be a linear transformation. Then if $\ker(T) = 0$, T is one-to-one.
11) **T-F** For any vectors $\mathbf{v_1}, \mathbf{v_2}$ in $\mathbb{R}^5$, $Span(v_1, v_2)$ is a vector subspace of $\mathbb{R}^5$.
12) **T-F** For any $n \times n$ matrix A, col(A) and null(A) are vector subspaces of $\mathbb{R}^n$.

    1) **False**: the null space of an $m \times n$ matrix is in $\mathbb{R}^n$.
    2) **True**: the column space of an $m \times n$ matrix is in $\mathbb{R}^m$.
    3) **False**: not necessarily. If the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for some nonzero $\mathbf{b}$, then that means that $\text{Col}(A) \neq \mathbb{R}^n$. But there can still be a nonzero element of the column space. For example, take $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
    4) **True**: If $\ker(T) = \{\mathbf{0}\}$, then T is an invertible linear transformation (i.e. represented by an invertible 3×3 matrix), and so it's column space must

span the whole of $\mathbb{R}^3$.

5) **True:** Let $\mathbf{x}$ be any vector. Then $A\mathbf{x} \in \text{Col}(A)$, so $A\mathbf{x} \in \text{Null}(A)$. But this means that $A(A\mathbf{x}) = 0$, so $A^2(\mathbf{x}) = 0$. So $\mathbf{x} \in \text{Null}(A^2)$. Since we can do this for any vector $\mathbf{x}$, it follows that $\text{Null}(A^2) = \mathbb{R}^n$.

6) **False:** for example, x^2 cannot be written as a linear combination of 1 and x. Rather, the set $\{1, x, x^2\}$ is a basis for P_2.

7) **True:** computation shows that $A\mathbf{v} = 0$.

8) **False:** the vector $\mathbf{v}$ is not a scalar multiple of the first column of A.

9) **False:** Take for example I and $-I$. Both of these are in the set of invertible matrices, however their sum is not: $I + (-I) = 0$.

10) **True:** Suppose that $\ker(T) = 0$. Suppose that there exist two vectors $x, y \in \mathbb{R}^n$ such that $T(x) = T(y)$. By linearity, this says that $T(x) - T(y) = T(x - y) = 0$. Since $\ker(T) = 0$, this implies that $x - y = 0$, so $x = y$. In other words, T is one-to-one.

11) **True.** The span of any set of vectors is by construction a vector subspace.

12) **True.** The column space and null space of a matrix is by construction a vector subspace. In particular, if we have an $n \times n$ matrix, then the column space and nullspace are both vector subspaces of $\mathbb{R}^n$.

Sections 5.1-5.5.

1) **T-F** If A is invertible, then A is diagonalizable.
2) **T-F** If A is diagonalizable, then there is a basis of $\mathbb{R}^n$ $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$, where each $\mathbf{v}_i$ is an eigenvector of A.
3) **T-F** A is diagonalizable if A has n not necessarily distinct eigenvalues.
4) **T-F** If A can be diagonalized as $A = PDP^{-1}$ for some diagonal matrix D and invertible matrix P, then D is unique.
5) **T-F** If A can be diagonalized as $A = PDP^{-1}$ for some diagonal matrix D and invertible matrix P, then P is unique.
6) **T-F** If M is **not** diagonalizable then it has an eigenvalue of algebraic multiplicity ≥ 2.
7) **T-F** If M has an eigenvalue of algebraic multiplicity ≥ 2 then it is **not** diagonalizable.
8) **T-F** Every M has n eigenvalues (counting algebraic multiplicity).
9) **T-F** 0 is an eigenvalue of M if and only if M is **not** invertible.
10) **T-F** M is invertible if and only if 0 is **not** an eigenvalue of M.
11) **T-F** If 1 is an eigenvalue of M and of M' then 1 is an eigenvalue of $M \cdot M'$.
12) **T-F** If M and M' are similar, then they have the same eigenvalues.
13) **T-F** M is invertible if and only if its eigenvectors span $\mathbb{R}^n$.
14) **T-F** If M is invertible then it is diagonalizable.
15) **T-F** If M is diagonalizable then it is invertible.

1) **False:** take e.g. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
2) **True:** If A is diagonalizable, then it has n linearly independent eigenvectors. But n linearly independent eigenvectors span $\mathbb{R}^n$, and so form a basis for $\mathbb{R}^n$.
3) **False:** All matrices have n not necessarily distinct eigenvalues. The example from two questions ago works as a counterexample here.

4) **False:** While the matrix D is always given by the eigenvalues of A, the order in which they appear on the diagonal is not unique.

5) **False:** While the matrix P is always given by the eigenvectors of A, an eigenvector is only unique up to a choice of basis for the associated eigenspace, which is not unique.

6) **True:** The only way to have a non-diagonalizable matrix is for the algebraic multiplicty of an eigenvalue to be greater than its geometric multiplicty. Since the geometric multiplicty is at least 1, this requires the algebraic multiplicty to be ≥ 2

7) **False**. Take the identity matrix - it has an eigenvalue 1 with algebraic multiplicity 2, but it is trivially diagonalizable.

8) **True**. Every $n \times n$ matrix has n eigenvalues, corresponding to the solutions of the characteristic equation. Note that some of these eigenvalues may be complex, but they will always appear in pairs as complex conjugates.

9) True. If a matrix has 0 as an eigenvalue, it has determinant 0.

10) **True.** This is a stronger statement than the previous one: a matrix has determinant 0 if and only if 0 is an eigenvalue of the matrix. Remember that the determinant of a matrix is always equal to the product of its eigenvalues.

11) **False.** Take $M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $M' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Both M and M' have 1 as an eigenvalue, but their product is $MM' = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which only has eigenvalue 0.

12) **True.** Similar matrices always have the same eigenvalues.

13) **False.** Take for example the zero matrix as a counterexample.

14) **False**. Take the upper triangular matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ as a counterexample.

15) **False.** Take for example the zero matrix as a counterexample.

Sections 6.1-6.7.

1) **T-F** If $\mathbf{u}$ and $\mathbf{v}$ are both orthogonal to $\mathbf{w}$, then they are orthogonal to each other: $\mathbf{u}$ is orthogonal to $\mathbf{v}$.

2) **T-F** For scalar $c \in \mathbb{R}$ and vector $\mathbf{v} \in \mathbb{R}^n$, $\|c\mathbf{v}\| = c\|\mathbf{v}\|$.

3) **T-F** If A is an matrix, with $\mathbf{v} = A^T\mathbf{w}$, $A\mathbf{u} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{v} = 0$.

4) **T-F** Given $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} \cdot \mathbf{v} \geq 0$.

5) **T-F** Every linearly independent set in $\mathbb{R}^n$ is an orthogonal set.

6) **T-F** Every orthogonal set in $\mathbb{R}^n$ is linearly independent.

7) **T-F** Given a nonzero scalar $c \in \mathbb{R}$, the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$ is the same as the orthogonal projection of $\mathbf{y}$ onto $c\mathbf{u}$.

8) **T-F** If $\mathbf{z}$ is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$, and $W = \operatorname{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, then $\mathbf{z} \in W^\perp$.

9) **T-F** If $V = \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and $\mathbf{x} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3$, then $\operatorname{proj}_V\mathbf{x} = \mathbf{x}$.

10) **T-F** Suppose $v_1, v_2, v_3 \in \mathbb{R}^n$ are linearly independent and $W = \operatorname{span}(v_1, v_2, v_3)$. Let (x_1, x_2, x_3) be an orthogonal nonzero set in W. Then (x_1, x_2, x_3) is a basis for W.

11) **T-F** If $x \in \mathbb{R}^n$ is not in the subspace V, then $x - \operatorname{proj}_V x \neq 0$.

12) **T-F** The list of vectors $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is the only orthonormal basis of $\mathbb{R}^2$.

13) **T-F** Let A be a $m \times n$ matrix with linearly independent columns. Apply Gram-Schmidt on the column vectors of A and have these vectors be the columns of another matrix B. Then $\operatorname{Col} A = \operatorname{Col} B$.

14) **T-F** If U is an $n \times p$ matrix with orthonormal columns, then $UU^T x = x$ for all $x \in \mathbb{R}^n$.

15) **T-F** A least-squares solution of $Ax = b$ is a vector $\hat{x}$ such that $||b - Ax|| \leq ||b - A\hat{x}||$ for all $x \in \mathbb{R}^n$.

16) **T-F** For any two vectors v and w in an inner product space, $\|v + w\| \leq \|v\| + \|w\|$.

17) **T-F** For any two vectors v and w in an inner product space, the following holds: $|\langle v, w\rangle| = \|v\|\|w\|$.

18) **T-F** Given a $m \times n$ matrix A and a vector $b \in \mathbb{R}^m$, there is always a unique least-squares solution $\hat{x} \in \mathbb{R}^n$.

19) **T-F** Let $W \subseteq V$ be a vector subspace of V. If $\mathbf{v} \in \text{span}(W)$, then $\text{proj}_W(\mathbf{v}) = \mathbf{v}$.

20) **T-F** Let $W \subseteq V$ be a vector subspace of V. If $\text{proj}_W(\mathbf{v}) = \mathbf{v}$, then $\mathbf{v} \in \text{span}(W)$.

21) **T-F** If $\{\mathbf{u}, \mathbf{v}\}$ is orthogonal and $\{\mathbf{v}, \mathbf{w}\}$ is orthogonal, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is orthogonal.

22) **T-F** If $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

23) **T-F** $\dim(W) = \dim(W^\perp)$.

24) **T-F** Let $\mathcal{A}$ be a set of nonzero vectors contained in some vector space V. Let $\mathcal{B}$ be the outcome of performing Gram-Schmidt to $\mathcal{A}$. Then $\text{span}(\mathcal{B}) = \text{span}(\mathcal{A})$.

25) **T-F** Let A be an $m \times n$ matrix. The least squares solution of $A\mathbf{x} = \mathbf{b}$ is the vector in $\mathbb{R}^m$ that is the shortest distance from $\mathbf{b}$.

26) **T-F** If $\mathbf{u}, \mathbf{v}$ are parallel vectors in $(V, \langle\cdot,\cdot\rangle)$, then $\langle \mathbf{u}, \mathbf{v}\rangle = \|\mathbf{u}\|\,\|\mathbf{v}\|$.

27) **T-F** Let V be a vector space, and let I_1 and I_2 be two inner products on V. If $\mathbf{u} \perp \mathbf{v}$ with respect to I_1, then $\mathbf{u} \perp \mathbf{v}$ with respect to I_2.

28) **T-F** Given an $m \times n$ matrix A, the matrix equation $A^T A\mathbf{x} = A^T\mathbf{b}$ is always consistent.

29) **T-F** Every basis $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ of $\mathbb{R}^3$ is orthogonal.

30) **T-F** For every n, the orthogonal complement of a line in $\mathbb{R}^n$ is also a line.

31) **T-F** For any vectors $\mathbf{u_1}, \mathbf{u_2}$ such that $W = Span\{\mathbf{u_1}, \mathbf{u_2}\}$ is a plane, the orthogonal projection of any vector $\mathbf{v}$ onto W is given by $proj_W(\mathbf{v}) = proj_{\mathbf{u_1}}(\mathbf{v}) + proj_{\mathbf{u_2}}(\mathbf{v})$.

32) **T-F** For any set of three vectors $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ the Gram-Schmidt algorithm will produce three **non-zero** orthogonal vectors $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$.

33) **T-F** For any 3×2 matrix A (3 rows, 2 cols) the equation $A\mathbf{x} = \mathbf{b}$ has a unique least squares solution for any $\mathbf{b}$ in $\mathbb{R}^3$.

34) **T-F** The only inner product is the dot product.

1) **False.** For a counterexample, take $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Then $\mathbf{u} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w} = 0$, but $\mathbf{u} \cdot \mathbf{v} = 5 \neq 0$.

2) **False.** Let $c = -1, \mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Then $\|c\mathbf{v}\| = \sqrt{(-3)^2 + (-4)^2} = 5$, but $\|\mathbf{v}\| = \sqrt{3^2 + 4^2} = 5$, so that $c\|\mathbf{v}\| = -5 \neq 5 = \| - c\mathbf{v}\|$.

3) **True.** The way that $\mathbf{v}$ was defined means that it is in the row space of A, and $\mathbf{u}$ is in the null space of A. From the text we know that $(\text{Row } A)^\perp = \text{Nul}A$, so that $\mathbf{u} \cdot \mathbf{v} = 0$.

4) **True.** $\mathbf{v} \cdot \mathbf{v} = \|v\|^2 \geq 0$.

5) **False.** For example, take the set $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^2$. This set is linearly independent, but $\mathbf{v}_1 \cdot \mathbf{v}_2 = 1 \neq 0$.

6) **False.** Every orthogonal set in $\mathbb{R}^n$ containing *only* nonzero vectors is a linearly independent set. However, there is no restriction requiring us to not have $\mathbf{0}$ as a vector in our orthogonal set.

7) **True.** To see this, note that by definition

$$\text{proj}_{\mathbf{u}}\mathbf{y} = \frac{\mathbf{u} \cdot \mathbf{y}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u}.$$

On the other hand,

$$\text{proj}_{c\mathbf{u}}\mathbf{y} = \frac{c\mathbf{u} \cdot \mathbf{y}}{c\mathbf{u} \cdot c\mathbf{u}}c\mathbf{u} = \frac{c^2}{c^2}\frac{\mathbf{u} \cdot \mathbf{y}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{y}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u},$$

so that $\text{proj}_{\mathbf{u}}\mathbf{y} = \text{proj}_{c\mathbf{u}}\mathbf{y}$.

8) **True.** For any $\mathbf{w} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 \in W$, by linearity of the dot product we have that $\mathbf{z} \cdot \mathbf{w} = c_1\mathbf{z} \cdot \mathbf{u}_1 + c_2\mathbf{z} \cdot \mathbf{u}_2 = c_1 \times 0 + c_2 \times 0 = 0$.

9) **True.** The vector $\mathbf{x}$ already lives in V, and so its projection is trivial.

10) **True.** Because (x_1, x_2, x_3) is orthogonal and nonzero, the list is linearly independent. Then it will be span W and be a basis for W.

11) **True.** Because x is not in V, then $\text{proj}_V x \neq x$.

12) **False.** We can create an orthogonal basis of $\mathbb{R}^2$ by choosing any 2 linearly independent vectors then applying Gram-Schmidt to orthogonalize, then normalize the vectors.

    Example:

$$v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

13) **True.** Similar logic to part (a), consider $\text{Col}\, A$ as the space spanned by the columns of A, then applying Gram Schmidt to its columns will provide an orthogonal basis for $\text{Col}\, A$. So the columns of B will also span $\text{Col}\, A$.

14) **False.** If we let the columns of U be a basis for W, then $UU^Tx = \text{proj}_W x$.

15) **False.** the inequality should be reversed, because $\hat{x}$ makes $A\hat{x}$ a better approximation for b than Ax.

16) **True.** this is the triangle inequality. A proof is in the book.

17) **False.** The equality should be replaced with $\leq$, which is the Cauchy-Schwarz inequality. A counterexample to the statement comes from taking $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in $\mathbb{R}^2$.

18) **False.** This is only true if A^TA is invertible (which is equivalent to $Nul(A) = 0$), in which case the unique solution is

$$\hat{x} = (A^TA)^{-1}A^Tb.$$

    A counterexample to the statement comes from taking A to be the zero matrix and b to be the zero vector; then any $\hat{x} \in \mathbb{R}^n$ is a least squares solution.

19) **True.** The projection of any vector onto a subspace containing that vector is trivial.

20) **True.** The orthogonal projection of $\mathbf{v}$ onto $\mathbf{w}$ is always a scalar multiple of $\mathbf{w}$, so by definition $\mathbf{v} \in \text{span}(\mathbf{w})$.

21) **False.** Take for example the set

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

22) **True.** This is a restatement of Pythagoras' theorem, and can be found in the book.

23) **False.** In fact, for $W \subseteq V$, $\dim(W) + \dim(W^{\perp}) = n$, where $n = \dim(V)$. So for example, any 1-dimensional subspace of $\mathbb{R}^3$ does the trick as a counterexample!

24) **True.** Gram Schmidt changes the vectors in the set, but not their span.

25) **False.** The least squares solution $\widehat{\mathbf{x}}$ is an approximation of a solution $\mathbf{x}$, not an approximation of $\mathbf{b}$. In other words, it is a vector $\widehat{\mathbf{x}}$ that minimises the distance between $A\mathbf{x}$ and $\mathbf{b}$.

26) **False.** If $\mathbf{u}$ and $\mathbf{v}$ are parallel, then $|\langle \mathbf{u}, \mathbf{v} \rangle| = \|\mathbf{u}\| \|\mathbf{v}\|$, but the crucial part of this is the absolute value. For example, take $\mathbf{u} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $\langle \mathbf{u}, \mathbf{v} \rangle = -2$, but $\|\mathbf{u}\| \|\mathbf{v}\| = \sqrt{2}\sqrt{2} = 2$. Obviously $-2 \neq 2$.

27) **False.** Orthogonality is **not** an intrinsic property of the underlying vector space, but very much so depends on the choice of inner product.

28) **True.** This is the foundational observation in the proof of the formula for calculating least square solutions. If the equation $A\mathbf{x} = \mathbf{b}$ is consistent, then this follows trivially. In the case that $A\mathbf{x} = \mathbf{b}$ is inconsistent, then any least squares solution to the system $A\mathbf{x} = \mathbf{b}$ is a solution to the system $A^T A\mathbf{x} = A^T\mathbf{b}$ (the normal equations), and so it is consistent.

29) **False.** The vectors $[1, 0, 0]^T, [1, 1, 0]^T$ and $[1, 1, 1]^T$ form a basis of $\mathbb{R}^3$ that is not orthogonal.

30) **False.** The orthogonal complement of a line in $\mathbb{R}^n$ is only a line when $n = 2$. When $n = 3$, for example, the orthogonal complement of a line is a plane.

31) **False.** This only works when your vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal.

32) **False**. You could start with one vector being $\mathbf{0}$, which would make one of the vectors at the end of the Gram-Schmidt process $\mathbf{0}$.

33) **False**. **False:** Take for example the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Then the orthogonal projection of $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ onto $col(A)$ is the vector $\hat{b} = \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix}$. The equation $A\hat{x} = \hat{b}$ is under determined since the columns of A are parallel, so there are multiple possible least squares solutions. In general, there are multiple least squares solutions whenever the columns of A are linearly dependent.

34) **False.** There are tons of other inner products! For example, the inner product $\int_0^1 f(x)g(x)\ dx$ on $C[0, 1]$ is an inner product that is certainly not the dot product.

Sections 7.1/5.7.

1) **T-F** If A is a 3×3 diagonalizable matrix of rank 1, then the dimension of the space of solutions to $x'(t) = Ax(t)$ of the form $x(t) = v$ where $v \in \mathbb{R}^3$ is a constant vector is 2.

2) **T-F** Suppose A is a 2×2 matrix, $b_1, b_2 \in \mathbb{R}^2$ vectors spanning $\mathbb{R}^2$. If $x_i(t)$ is a particular solution to $x'(t) = Ax(t) + b_i$ for $i = 1, 2$, then there are nonzero constants such that $c_1 x_1(t) + c_2 x(t)$ is a solution to $x'(t) = Ax(t)$.

3) **T-F** Let A be a square matrix. We always have that $x'(t) = Ax(t)$ has a solution.

    1) **True.** The options for v are precisely the vectors in the null space of A, which by the Rank-Nullity Theorem has dimension 2.

    2) **False.** The difference of two solutions to the same inhomogenous equation is a solution to the corresponding homogenous equation, if the vectors b_i are linearly independent, such a linear combination does not necessarily exist.

    3) **True.** This is the existence theorem for systems of ordinary differential equations. Also, we can just take $x(t) = 0$ since the equation is homogeneous.