MATH 54 Quiz 7 **Name:**

Please write legibly and explain your work clearly. Answers without explanations may receive less (or no) credit.

Problem 1. Suppose $A = \begin{bmatrix} 0 & 1 & -3 & 4 & -1 & 9 \\ 0 & -2 & 6 & -6 & -1 & -10 \\ 0 & -3 & 9 & -6 & -6 & -3 \\ 0 & 3 & -9 & 4 & 9 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and assume that A and B are row equivalent.

(a) What is a basis for Null A?

Solution: find these by setting 1 of the 3 free variables to 1 at a time (and setting the others to zero) and solving the resulting system of equations using B.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 0 \\ 3/2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(b) What is a basis for Col A?

Solution: we look at B to find the pivot columns. We then take those columns from A.

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 9 \\ -10 \\ -3 \\ 0 \end{bmatrix} \right\}$$

(c) Is the transformation $T : \mathbb{R}^6 \to \mathbb{R}^4$ given by $T(\mathbf{x}) = A\mathbf{x}$ one-to-one? **No**, the first column has no pivot. Alternatively, we have a nontrivial null space by part (a).

(d) Is the transformation $T : \mathbb{R}^6 \to \mathbb{R}^4$ given by $T(\mathbf{x}) = A\mathbf{x}$ onto? **No**, the last row has no pivot. Alternatively, the column space from part (b) has only 3 vectors, so it cannot span all of $\mathbb{R}^4$.

Problem 2. Find an orthogonal basis of $\mathbb{R}^3$ that includes the vector $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

Solution: first, let's create a (non-orthogonal) basis. We will use the basis

$$\mathbf{u} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

To see that this is a basis, simply notice that the matrix $\begin{bmatrix} \mathbf{u} & \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$ is lower diagonal, and all diagonal entries are nonzero. This means it has a nonzero determinant, and nonzero determinant means basis.

Let's first make $\mathbf{v}_1$ orthogonal to $\mathbf{u}$.

$$\mathbf{u}_1 = \mathbf{v}_1 - \operatorname{proj}_{\mathbf{u}} \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ -1/3 \end{bmatrix}$$

and now let's make $\mathbf{v}_2$ orthogonal to $\mathbf{u}$ and $\mathbf{u}_1$.

$$\mathbf{u}_2 = \mathbf{v}_2 - \operatorname{proj}_{\mathbf{u}} \mathbf{v}_2 - \operatorname{proj}_{\mathbf{u}_1} \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \frac{-1/3}{2/3} \begin{bmatrix} 1/3 \\ 2/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}$$

Problem 3. Let V be continuous functions on $[0, 2\pi]$ with the inner product

$$\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)\mathrm{d}x.$$

(a) Show that $\{1, \cos(x)\}$ is an orthogonal set.

Solution: we simply need to check that the inner product is zero.

$$\langle 1, \cos(x) \rangle = \int_0^{2\pi} 1\cos(x)\mathrm{d}x = \sin(x)\Big|_0^{2\pi} = 0$$

(b) Find the orthogonal projection of $\sin^2(x)$ onto the span of $\{1, \cos(x)\}$.

Solution: Let's collect a few integrals.

$$\langle 1, 1 \rangle = \int_0^{2\pi} 1^2\mathrm{d}x = 2\pi$$

$$\langle \sin^2, 1 \rangle = \int_0^{2\pi} \sin^2(x)\mathrm{d}x = \frac{1}{2}\int_0^{2\pi} 1 - \cos(2x)\mathrm{d}x = \pi - \frac{1}{4}\sin(2x)\Big|_0^{2\pi} = \pi$$

$$\langle \cos, \cos \rangle = \int_0^{2\pi} \cos^2(x)\mathrm{d}x = \int_0^{2\pi} 1 - \sin^2(x)\mathrm{d}x = 2\pi - \pi = \pi$$

$$\langle \sin^2, \cos \rangle = \int_0^{2\pi} \sin^2(x)\cos(x)\mathrm{d}x = \int u^2\mathrm{d}u = \frac{u^3}{3} = \frac{\sin^3(x)}{3}\Big|_0^{2\pi} = 0$$

Now we can compute the desired projection.

$$\frac{\langle \sin^2, 1 \rangle}{\langle 1, 1 \rangle}(1) + \frac{\langle \sin^2, \cos \rangle}{\langle \cos, \cos \rangle}(\cos(x)) = \frac{\pi}{2\pi}(1) + \frac{0}{\pi}\cos(x) = \frac{1}{2}$$

[*Possibly useful hints: u-sub using* $u = \sin(x)$*;* $\cos(2x) = 1 - 2\sin^2(x)$]