MATH 54 Quiz 6 **Name:**

Please write legibly and explain your work clearly. Answers without explanations may receive less (or no) credit.

Answer any 3 of the following 4 questions. Select the 3 questions you would like me to grade.

Problem 1. Suppose U is an $n \times n$ matrix so that $(U\mathbf{x}) \cdot (U\mathbf{x}) < 0$ for some $\mathbf{x} \in \mathbb{R}^n$. Prove that U is not an orthogonal matrix. State any theorems that you use and explain how they are used. [*Don't state the name of a theorem, state what the theorem says. If you are unsure what is/isn't a theorem, a good rule would be to state any facts that you use.*]

Solution: the originally intended idea was:

1. Fact 1: If U is orthogonal then $(U\mathbf{x}) \cdot (U\mathbf{x}) = \mathbf{x} \cdot \mathbf{x}$.
2. Fact 2: $\mathbf{x} \cdot \mathbf{x} \geq 0$.

Putting these facts together, if U was orthogonal, then $(U\mathbf{x}) \cdot (U\mathbf{x}) = \mathbf{x} \cdot \mathbf{x} \geq 0$. Thus the U cannot be orthogonal.

However, I made a bit of a mistake. There is an observation that you could use to basically avoid the whole problem, and it may have been confusing. By fact (2), $(U\mathbf{x}) \cdot (U\mathbf{x}) \geq 0$. Thus U doesn't even exist, so it is certainly not an orthogonal matrix.

A better problem would have been: $(U\mathbf{x}) \cdot (U\mathbf{x}) = 0$ for some $\mathbf{x} \neq 0$ in $\mathbb{R}^n$.

Problem 2. Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$. Given that $\{\mathbf{a}_1, \mathbf{a}_2\}$ is an **orthonormal basis** of $\text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, find $\mathbf{a}_3$ so that A is an **orthogonal** matrix.

Solution: An orthogonal matrix must have orthonormal columns. Thus we need to find a vector in the orthogonal complement to the space spanned by the first two columns. Once we have that vector, then we will make it unit length. To find the vector, we simply compute

$$\text{Null} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \text{Null} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \text{Null} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

we conclude that

$$\mathbf{a}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} / (\sqrt{(-1)^2 + 0^2 + 1^2}) = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

Problem 3. Imagine you are living on a slope in 2 dimensions, specifically on the line $y = -x/2$. There is a bird in the sky at the point $(1, 20)$. You want to take the best picture of the bird, which will happen when you are closest to the bird. Where should you stand to take the picture? [*Hint: can you find a basis for the subspace* $\{y = -x/2\}$*?*]

Solution: First we need to know what subspace we want to project on. Note that the line $y = -x/2$ is spanned by $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ since $1 = (-2)/2$ is a point on the line. Thus the point on the line closest to the bird is given by the projection

$$\text{proj}_{\mathbf{v}} \begin{bmatrix} 1 \\ 20 \end{bmatrix} = \left(\frac{\begin{bmatrix} 1 \\ 20 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}}{(-2)^1 + 1^2} \right) \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \frac{18}{5} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

and so the point is $(-36/5, 18/5)$.

Problem 4. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$ be two bases of $\mathbb{R}^2$. Given $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ find $[\mathbf{x}]_{\mathcal{C}}$.

Solution: let $P_{\mathcal{C}} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $P_{\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. We need to solve the system

$$P_{\mathcal{C}}[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{B}}[\mathbf{x}]_B \qquad \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

which we now recognize as a row-reduction problem

$$\left[\begin{array}{cc|c} 1 & 2 & -1 \\ 2 & 3 & -3 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & -1 & -1 \end{array}\right]$$

and now we read that the solution is $x_2 = 1$ and $x_1 + 2x_2 = -1 \implies x_1 = -3$. Thus $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.