MATH 54 Quiz 5 **Name:**

Please write legibly and explain your work clearly. Answers without explanations may receive less (or no) credit.

Problem 1. True or False. In the following questions, M a and N are $n \times n$ matrices.

T—F If M has an eigenvalue of algebraic multiplicity ≥ 2 then it is **not** diagonalizable.

False. For example, the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is diagonalizable and the only eigenvalue is 2, with algebraic multiplicity 2.

T—F If M and N are similar, then they have the same eigenvalues.

True. This is a theorem.

T—F M is invertible if and only if its eigenvectors span $\mathbb{R}^n$.

False. Take $M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Its eigenvectors are $\mathbf{e}_1$ with eigenvalue 1 and $\mathbf{e}_2$ with eigenvalue zero. It is clearly noninvertible (zero determinant, not row equivalent to identity, any other condition from invertible matrix theorem).

T—F If M is invertible then it is diagonalizable.

False. Take the matrix of a shear transformation, for example $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. It can be shown that the matrix is not diagonalizable (the only eigenvalue is 1, but the 1-eigenspace is only spanned by one vector), but it is invertible, with $M^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

T—F If M and N both have an eigenvalue 1, then one of the eigenvalues of MN is 1.

False. Take $M = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $N = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Then each has an eigenvalue of 1, but $MN = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ only has 2 as an eigenvalue.

Problem 2. Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be a linear transformation defined by

$$T(p) = p(1) + p(0)t + p(-1)t^2.$$

With respect to the basis $\{1, t, t^2\}$ of $\mathbb{P}_2$, what is the matrix for T?

Solution: Compute.

$$\begin{aligned}
T(b_1) &= T(1) = 1(1) + 1(t) + 1(t^2) = 1b_1 + 1b_2 + 1b_3 \\
T(b_2) &= T(t) = 1(1) + 0(t) - 1(t^2) = 1b_1 - 1b_3 \\
T(b_3) &= T(t^2) = 1(1) + 0(t) + 1(t^2) = 1b_1 + 1b_3
\end{aligned}$$

. Putting this into a matrix, we get

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

With respect to the basis $\{1, t^2, 2 + t + t^2\}$ of $\mathbb{P}_2$, what is the matrix for T?

$$\begin{aligned}
T(b_1) &= T(t) = 1(1) + 0(t) - 1(t^2) = 1(2 + t + t^2) - 1(1) = b_3 - b_1 \\
T(b_2) &= T(t^2) = 1(1) + 0(t) + 1(t^2) = 1b_1 + 1b_2 \\
T(b_3) &= T(2 + t + t^2) = 4(1) + 2(t) + 2(t^2) = 2(2 + t + t^2) = 2b_3
\end{aligned}$$

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$$[T]_{\mathcal{B}} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Problem 3. Let

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Diagonalize A (i.e. find the invertible matrix P and the diagonal matrix D so that $A = PDP^{-1}$).

Solution: We read off the eigenvalues of A: 3,2,2. Next we compute a basis for each eigenspace

$$\ker[A - 3I] = \ker \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

and

$$\ker[A - 2I] = \ker \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

Thus we have found $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. To verify that these work, we will check that $AP = PD$.

$$PD = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

and

$$AP = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

so we have indeed diagonalized A.