MATH 54 Quiz 4
Solutions

Problem 1 (5 points). Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by

$$T\left(\begin{bmatrix}3\\1\end{bmatrix}\right) = \begin{bmatrix}5\\3\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}2\\-2\end{bmatrix}.$$

(a) What is the standard matrix representing T? (No partial credit for this part)

Solution: We compute

$$T\left(\begin{bmatrix}5\\0\end{bmatrix}\right) = T\left(2\begin{bmatrix}3\\1\end{bmatrix} - \begin{bmatrix}1\\2\end{bmatrix}\right) = 2\begin{bmatrix}5\\3\end{bmatrix} - \begin{bmatrix}2\\-2\end{bmatrix} = \begin{bmatrix}8\\8\end{bmatrix} \implies T(\vec{\mathbf{e}}_1) = \begin{bmatrix}8/5\\8/5\end{bmatrix}$$

and

$$T\left(\begin{bmatrix}0\\5\end{bmatrix}\right) = T\left(3\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}3\\1\end{bmatrix}\right) = 3\begin{bmatrix}2\\-2\end{bmatrix} - \begin{bmatrix}5\\3\end{bmatrix} = \begin{bmatrix}1\\-9\end{bmatrix} \implies T(\vec{\mathbf{e}}_2) = \begin{bmatrix}1/5\\-9/5\end{bmatrix}$$

from which we can determine that the matrix representing T is

$$A = \begin{bmatrix}8/5 & 1/5\\8/5 & -9/5\end{bmatrix}$$

Problem 2 (5 points).

(a) Let V be a vector space. Define what it means for H to be a subspace of V.

Solution: H is a subspace of V if and only if

i. $\mathbf{0} \in H$.

ii. For any $\mathbf{u}, \mathbf{v} \in H$, $\mathbf{u} + \mathbf{v} \in H$.

iii. For any $c \in \mathbb{R}$, $c\mathbf{u} \in H$.

(b) Let $V = \mathbb{R}^3$ and let $H \subseteq V$ be the set of points whose coordinates sum up to 1 and the point $\mathbf{0}$ In mathematical notation,

$$H = \left\{ \vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 1 \text{ or } x_1 = x_2 = x_3 = 0 \right\}.$$

Show that H is not a subspace of V.

Solution: We know that the point $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is in H, since $1 + 0 + 0 = 1$. But $\vec{\mathbf{u}} + \vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ is not in H since $2+0+0 \neq 1$. Thus we have failed condition (ii) above.

Problem 3 (5 points). Let

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

Solution: First we row reduce

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Compute a basis for the null space of A. Explain why you have found a basis.

Solution: In the above row reduction, we notice that x_4 is the only free variable. Setting it to 1 yields the (only) basis element of the null space,

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

We know this is a basis because all elements of the kernel are uniquely determined by how we choose x_4; if we choose $x_4 = a$ then we get

$$\begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(b) Compute a basis for the column space of A. Explain why you have found a basis.

Solution: We need to find the largest linearly independent subset of the columns of A. To do this, we can just pick columns of A that correspond to pivot columns of the row reduced matrix. In our case, we could construct a basis using the first 3 columns,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

This is a basis because the span is the same as the span of all the columns of A and the vectors are linearly independent. To see this, note that the row reduction demonstrates that the last column is a linear combination of the first 3, so we exclude it from our basis set. It also demonstrates that the third column is NOT a linear combination of the first two, so we need the first 3 columns in our basis set.