**MATH 54 Quiz 3**

**Name:** Solutions **Student ID:**

*Please put your name and student ID above. You have 25 minutes to attempt the following problems. You may not use notes or electronic devices during this time. Please write legibly and explain your work clearly.*

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**Problem 1 (5 points).** List 5 equivalent conditions for a matrix to have an inverse.

For all of the following let $A$ be an $n \times n$ matrix (note it must be square!).

1. $A^T$ is invertible
2. The span of the columns of $A$ is all $\mathbb{R}^n$
3. the columns of $A$ are linearly independent
4. the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$
5. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = 0$
6. The associated transform $\mathbf{x} \to A\mathbf{x}$ is 1-1
7. The associated transform $\mathbf{x} \to A\mathbf{x}$ is onto

**Problem 2 (5 points).** True or False? (No justification required.)

(i) T — F If $A$ is a $4 \times 4$ matrix, then $\det(-A) = \det(A)$.

**True!** This is the same as multiplying each of the four rows by $-1$. So, the determinant is multiplied by $(-1)^4 = 1$.

(ii) T — F If the columns of a $3 \times 3$ matrix $A$ sum to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, then $A$ is invertible. **False!** Take for example the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(iii) T — F If $A$ and $B$ are square matrices, $\det(A + B^T) = \det(A + B)$. **False!** Take for example the matricies

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then,

$$\det A + B^T = \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 1 \qquad \det A + B = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = 0$$

(iv) T — F If the linear transformation $\mathbb{R}^n \to \mathbb{R}^n$ defined by $A$ is injective (one-to-one), then $A$ is surjective (onto). True! This is a consequence of the invertible matrix theorem

(v) T — F Suppose $A$ can be reduced to the identity by subtracting row 1 from row 2, then subtracting column 1 from column 3, then multiplying the second row by 2 and multiplying the first column by 1/4. Then $\det A = 2$. **True!** When we subtract multiples of rows or columns, we don't change the determinant, so the only steps we have to worry about are the multiplication of rows and columns by constants. In this case, we have that $2(1/4) \det A = \det I = 1 \implies \det A = 2$.

**Problem 3 (5 points).** Let

$$A = \begin{bmatrix} 3 & 3 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ 3 & 6 & -1 & 0 & 0 \\ 1 & 2 & 10 & 9 & 0 \\ 3 & 2 & 6 & 4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 5 & 6 & 10 \\ 3 & 4 & 7 & 2 & 9 \\ 2 & 5 & 6 & -4 & -6 \\ -2 & 4 & 6 & -7 & -7 \\ 1 & 1 & 0 & 0 & 6 \end{bmatrix}.$$

Assume B is invertible, and find $\det(BAB^{-1})$. Explicitly state any determinant rules you use.

We use the following properties:

$$\det(AB) = \det(A)\det(B) \quad \det(B^{-1}) = 1/\det(B)$$

So,

$$\det(BAB^{-1}) = \det(B)\det(A)\det(B^{-1}) = \frac{\det(B)\det(A)}{\det(B)} = \det(A)$$

(for those keeping track at home, $\det(B) = -3594$. Luckily, we don't have to calculate it!)

Now we must calculate $\det(A)$. We expand along the last column in each step

$$\begin{aligned}
\det(BAB^{-1}) = \det(A) &= \begin{vmatrix} 3 & 3 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ 3 & 6 & -1 & 0 & 0 \\ 1 & 2 & 10 & 9 & 0 \\ 3 & 2 & 6 & 4 & 1 \end{vmatrix} \\
&= 1 \cdot \begin{vmatrix} 3 & 3 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 3 & 6 & -1 & 0 \\ 1 & 2 & 10 & 9 \end{vmatrix} \\
&= 1 \cdot 9 \cdot \begin{vmatrix} 3 & 3 & 0 \\ 2 & 3 & 0 \\ 3 & 6 & -1 \end{vmatrix} \\
&= 1 \cdot 9 \cdot -1 \cdot \begin{vmatrix} 3 & 3 \\ 2 & 3 \end{vmatrix} \\
&= 1 \cdot 9 \cdot -1 \cdot (9 - 6) \\
&= \boxed{-27}
\end{aligned}$$