MATH 54 Quiz 2: Solutions

Problem 1 (5 points). Let A be as follows.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

(a) Compute A^{-1}.

Solution: We can use row reduction.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

and now we can conclude that the inverse matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

(b) What vector $\vec{\mathbf{x}}$ solves the equation

$$A\vec{\mathbf{x}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}?$$

Solution: We can simply compute

$$\vec{\mathbf{x}} = A^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b - a \\ c - b \end{bmatrix}$$

Problem 2 (5 points). True or False? (No justification required)

(i) If A and B are $n \times n$ matrices, then $AB = BA$.

False. For example, try $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(ii) Suppose $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ is a linearly dependent set of vectors in $\mathbb{R}^6$. Then $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\}$ is also a linearly dependent set.

False. For example, take $\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{\mathbf{v}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{\mathbf{v}}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

(iii) If A is a 2×3 matrix then A is never onto.

False. For example, the 2×3 matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is onto.

(iv) If A is an $n \times n$ square matrix and $A^\top A = 2I$, then A is invertible.

.**True.** The inverse of A is $\frac{1}{2}A^\top$. To verify this, we can observe that $\frac{1}{2}A^\top A = \frac{1}{2}(2I) = I$ and (to verify that it is also a right inverse, although this is not necessary for square matrices) we can compute $A(\frac{1}{2}A^\top) = \frac{1}{2}(AA^\top) = \frac{1}{2}(A^\top A)^\top = \frac{1}{2}(2I)^\top = I$.

(v) If A and B are invertible $n \times n$ square matrices then $(A + B)^{-1} = A^{-1} + B^{-1}$.

False. In fact, $A + B$ may not even be invertible! Try $A = I$ and $B = -I$

Problem 3 (5 points). Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$T\left(\begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_2 + x_3 \\ -x_3 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

(a) What is the standard matrix representing T?

Solution: We compute

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) - T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

from which we can determine that the matrix representing T is

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$