Sample Problems with Explanations (preparation for midterm)

Students from Math 54 sections 201 and 203

March 15, 2022

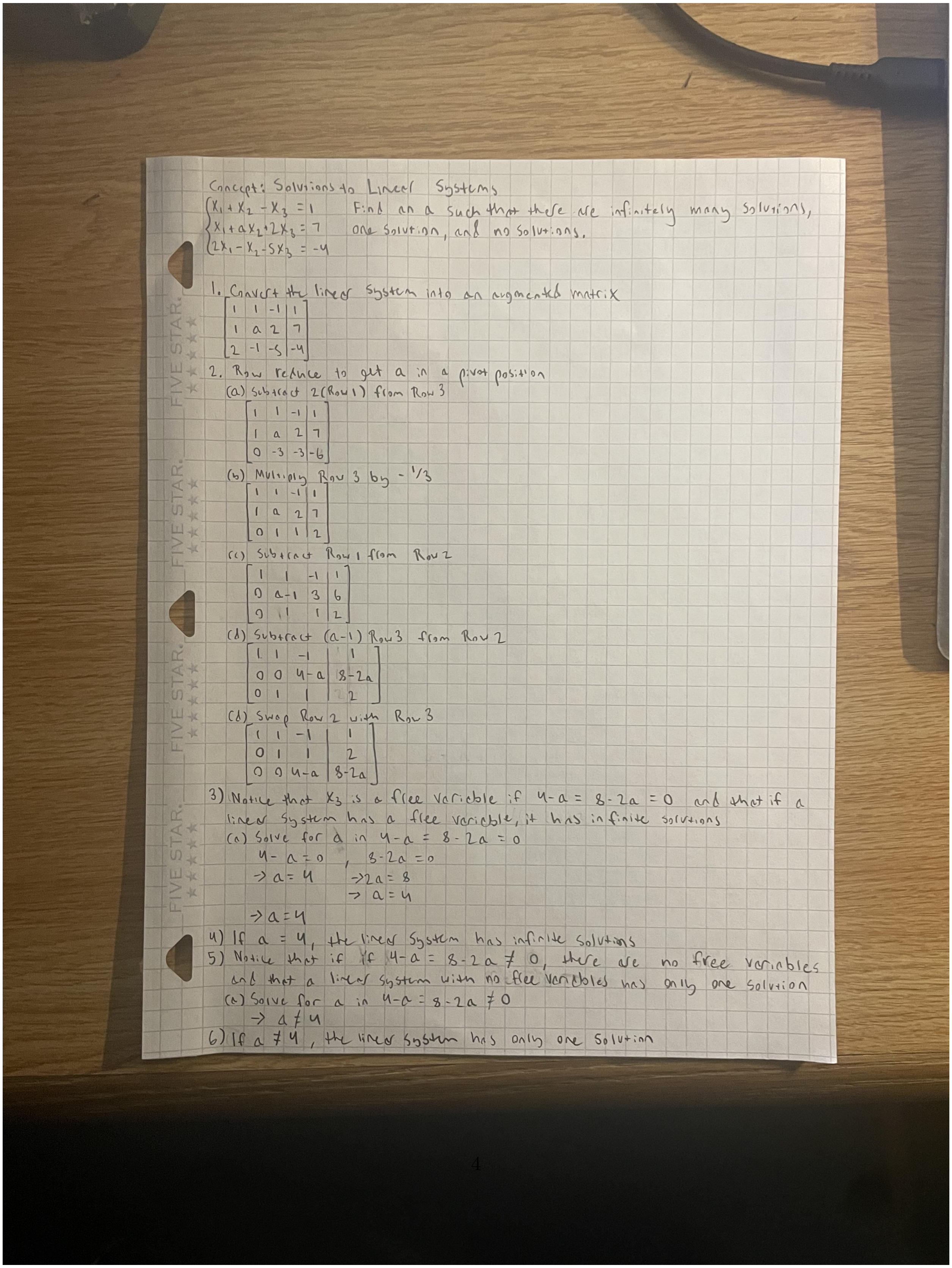
I have categorized the work you all did for easier reference. You should be able to click on the headings in the table of contents and be taken to the appropriate section. Some people's work fit into multiple categories. In this case I categorized based on what I thought was the major theme.

Credit to each and every one of you to contributing to this document!

Contents

1 Solving systems of linear equations (consistency, existence, uniqueness)

1.1 [Steven]



Concept: Solutions to Linear Systems

$$\begin{cases} x_1 + x_2 - x_3 = 1 \\ x_1 + a x_2 + 2x_3 = 7 \\ 2x_1 - x_2 - 5x_3 = -4 \end{cases}$$

Find an a such that there are infinitely many solutions, one solution, and no solutions.

1. Convert the liner system into an augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & a & 2 & 7 \\ 2 & -1 & -5 & -4 \end{array}\right]$$

2. Row reduce to get a in a pivot position

(a) Subtract 2(Row 1) from Row 3

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & a & 2 & 7 \\ 0 & -3 & -3 & -6 \end{array}\right]$$

(b) Multiply Row 3 by $-1/3$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & a & 2 & 7 \\ 0 & 1 & 1 & 2 \end{array}\right]$$

(c) Subtract Row 1 from Row 2

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & a-1 & 3 & 6 \\ 0 & 1 & 1 & 2 \end{array}\right]$$

(d) Subtract (a-1) Row 3 from Row 2

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 0 & 4-a & 8-2a \\ 0 & 1 & 1 & 2 \end{array}\right]$$

(d) Swap Row 2 with Row 3

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 4-a & 8-2a \end{array}\right]$$

3) Notice that x_3 is a free variable if $4-a = 8-2a = 0$ and that if a liner system has a free variable, it has infinite solutions

(a) Solve for a in $4-a = 8-2a = 0$

$4 - a = 0$, $8 - 2a = 0$

$\rightarrow a = 4$ $\rightarrow 2a = 8$

$\rightarrow a = 4$

$\rightarrow a = 4$

4) If $a = 4$, the liner system has infinite solutions

5) Notice that if if $4-a = 8-2a \neq 0$, there are no free variables and that a liner system with no free variables has only one solution

(a) Solve for a in $4-a = 8-2a \neq 0$

$\rightarrow a \neq 4$

6) If $a \neq 4$, the liner system has only one solution

7) Notice that if $4-a \neq 0$ while $8-2a=0$, then the linear system is inconsistent (it has no solutions)

(a) Find a such that $4-a \neq 0$, $8-2a=0$

(b) Notice that this is not possible because only $a=4$ allows $8-2a=0$, but $a=4$ does not allow $4-a \neq 0$

8) No a makes it so that there are no solutions

1.2 [Warren]

You are given the linear equations

$$x + 2y + az = 0$$
$$-x + z = 0$$
$$ax - y + z = 0$$

Find the values of a for which the system has a unique solutions; no solutions.

create system of matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & a & 0 \\ -1 & 0 & 1 & 0 \\ a & -1 & 1 & 0 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{ccc|c} 1 & 2 & a & 0 \\ 0 & 2 & 1+a & 0 \\ 0 & -2a-1 & 1-a^2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & a & 0 \\ 0 & 1 & \frac{1+a}{2} & 0 \\ 0 & -2a-1 & 1-a^2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & a & 0 \\ 0 & 1 & \frac{1+a}{2} & 0 \\ 0 & 0 & b & 0 \end{array}\right]$$

There are unique solutions for $a \neq 1$ because it would prove that $\_ = 0$.

There is inf. many solutions solutions if $a = -1$ b/c $0 = 0$ is always true

Because of this, there will never not be a solution

$$b = (1-a^2) - (-2a-1)\left(\frac{1+a}{2}\right)$$
$$0 = (1-a^2) + (2a+1)\left(\frac{1+a}{2}\right)$$
$$= 1 - a^2 + a^2 + \frac{3}{2}a + \frac{1}{2}$$
$$0 = \frac{3}{2}a + \frac{3}{2}$$
$$-\frac{3}{2} = \frac{3}{2}a$$
$$a = -1$$

2 Linear transformations

2.1 [Kwan]

Linear Transformations

A transformation (or mapping) T is linear if :

1. $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in the domain of T
2. $T(c\vec{u}) = cT(\vec{u})$ for all scalars c and all $\vec{u}$ in the domain of T

If T is a linear transformation, then

$$T(\vec{0}) = \vec{0} \quad \text{and} \quad T(c\vec{u}+d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$$

for all vectors $\vec{u}$ and $\vec{v}$ in the domain of T and all scalars c, d.

Ex. Define a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

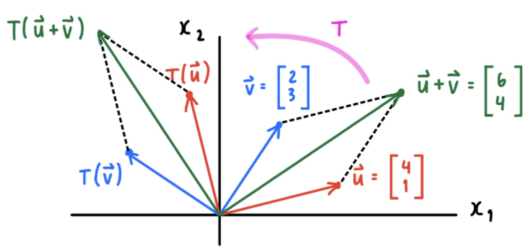
Find the images under T of $\vec{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and $\vec{u}+\vec{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

$$T(\vec{u}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} (0)(4) + (-1)(1) \\ (1)(4) + (0)(1) \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$T(\vec{v}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} (0)(2) + (-1)(3) \\ (1)(2) + (0)(3) \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$T(\vec{u}+\vec{v}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} (0)(6) + (-1)(4) \\ (1)(6) + (0)(4) \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$$

Note that $T(\vec{u}+\vec{v})$ is equal to $T(\vec{u}) + T(\vec{v})$ since the transformation T is linear



3 Standard matrix

3.1 [Anna M]

Say $T: \mathbb{R}^3 \to \mathbb{R}^3$ and

$$T\left(\begin{bmatrix}1\\3\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\1\\3\end{bmatrix}, \quad T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Find the matrix of this linear transformation.

$$\vec{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} = a\begin{bmatrix}1\\3\\1\end{bmatrix} + b\begin{bmatrix}0\\1\\1\end{bmatrix} + c\begin{bmatrix}1\\1\\0\end{bmatrix}$$

① $1 = a + c$ ② $0 = 3a + b + c$ ③ $0 = a + b$

subtract ① and ③ from ②

$-1 = a \rightarrow b = 1, \; c = 2$

$$\Rightarrow T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = -T\left(\begin{bmatrix}1\\3\\1\end{bmatrix}\right) + T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) + 2T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)$$

$$= -\begin{bmatrix}0\\1\\1\end{bmatrix} + \begin{bmatrix}2\\1\\3\end{bmatrix} + 2\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}2\\0\\4\end{bmatrix}$$

$$\vec{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} = a\begin{bmatrix}1\\3\\1\end{bmatrix} + b\begin{bmatrix}0\\1\\1\end{bmatrix} + c\begin{bmatrix}1\\1\\0\end{bmatrix}$$

① $0 = a + c$ ② $1 = 3a + b + c$ ③ $0 = a + b$

$1 = a \rightarrow b = -1, \; c = -1$

$$\Rightarrow T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\3\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)$$

$$= \begin{bmatrix}0\\1\\1\end{bmatrix} - \begin{bmatrix}2\\1\\3\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}-2\\0\\-3\end{bmatrix}$$

$$\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = a\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

① $0 = a + c$ ② $0 = 3a + b + c$ ③ $1 = a + b$

$-1 = a \rightarrow b = 2, \; c = 1$

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = -T\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}\right) + 2T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$$

$$= -\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$$

Transformation matrix $= \left[T(\vec{e}_1) \,|\, T(\vec{e}_2) \,|\, T(\vec{e}_3) \right]$

$$= \begin{bmatrix} 2 & -2 & 4 \\ 0 & 0 & 1 \\ 4 & -3 & 6 \end{bmatrix}$$

3.2 [Karlaine]

Karlaine Francisco
Example Problem + Solution

A linear transformation T from $\mathbb{R}^2 \to \mathbb{R}^3$ satisfies

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}6\\4\\9\end{bmatrix}, \quad T\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}0\\5\\0\end{bmatrix}$$

Write the standard matrix of T. Why is that the answer?

SOLUTION

A standard matrix for a linear transformation would just be a matrix, we can call A where every input x is in ($\in$) $\mathbb{R}^m$. [Let $T: \mathbb{R}^m \to \mathbb{R}^n$]

it would also have the dimensions $n \times m$, so in this case

$$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$ ← it would look like that.

instead of using dots(·) we can use variables to set up our standard matrix!

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$ ∴ then we can use our other definition of the standard matrix,

$$A = [T(e_1), T(e_2), \dots T(e_m)]$$

we can also use our lin. tranformation rules!

1. $T(x+y) = T(x) + T(y)$
2. $T(cx) = cTx$

$$\begin{bmatrix}6\\4\\9\end{bmatrix} = T\begin{pmatrix}1\\1\end{pmatrix} = A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}a & b\\c & d\\e & f\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}a+b\\c+d\\e+f\end{bmatrix} = \begin{bmatrix}6\\4\\9\end{bmatrix}$$

$a+b=6$
$c+d=4$
$e+f=9$

$$\begin{bmatrix}0\\5\\0\end{bmatrix} = T\begin{pmatrix}1\\-2\end{pmatrix} = A\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}a & b\\c & d\\e & f\end{bmatrix}\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}a-2b\\c-2d\\e-2f\end{bmatrix} = \begin{bmatrix}0\\5\\0\end{bmatrix}$$

↓ next page

Then, w/ regards to these $\underbrace{\text{six equations}}$

$$\begin{aligned} a+b&=6 & a-2b&=0 \\ c+d&=4 & c-2d&=5 \\ e+f&=9 & e-2f&=0 \end{aligned}$$

we find out what each variable is!

$a+b=6$
$\quad -b \quad -b$
$a=6-b$
$6-1b-2b=0$
$6-3b=0$
$-6 \qquad -6$
$\frac{-3b}{-3}=\frac{-6}{-3}$
$\boxed{b=2}$

$a+2=6$
$\boxed{a=4}$

$c+d=4$
$-c \quad -c$
$d=4-c$

$c-2(4-c)=5$
$1c-8+2c=5$
$-8+3c=5$
$+8 \qquad +8$
$3c=13$
$\boxed{c=13/3}$

$e+f=9$
$-e \quad -e$
$f=9-e$

$e-2(9-e)=0$
$e-18+2e=0$
$3e-18=0$
$+18 \quad +18$
$3e=18$
$\boxed{e=6}$

$f=9-6=3$
$\boxed{f=3}$

$\frac{3}{13}\cdot\frac{13}{3}-2d=5\cdot\frac{13}{3}$
$\frac{-2d}{-2}=\frac{65}{3}\Big/-2$
$\boxed{d=-1/3}$

Now that we have values for all of our variables we can put it into our standard matrix

$$\begin{bmatrix} 4 & 2 \\ 13/3 & -1/3 \\ 6 & 3 \end{bmatrix}$$

And to double check if its right: we multiply it w/ our 2 given vectors $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-2\end{bmatrix}$

$$\begin{bmatrix} 4 & 2 \\ 13/3 & -1/3 \\ 6 & 3 \end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix} 4+2 \\ 12/3=4 \\ 6+3 \end{bmatrix} = \begin{bmatrix}6\\4\\9\end{bmatrix} \checkmark$$

AND $\begin{bmatrix} 4 & 2 \\ 13/3 & -1/3 \\ 6 & 3 \end{bmatrix}\begin{bmatrix}1\\-2\end{bmatrix}$ cont..

cont.

$$\begin{bmatrix} 4 & 2 \\ \frac{13}{3} & -1/3 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{aligned} 4 + 2(-2) &= \\ \frac{13}{3} + -\frac{1}{3}(-2) &= \\ 6 + 3(-2) &= \end{aligned} \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} \checkmark$$

:)

$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 6 \\ 4 \\ 9 \end{bmatrix}$ & $T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$

have a std. matrix

$$\begin{bmatrix} 4 & 2 \\ 13/3 & -1/3 \\ 6 & 3 \end{bmatrix}$$!!

3.3 [Ezra]

5. (25 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation such that

$$T\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\t+1\\t+2\end{pmatrix}, \quad T\begin{pmatrix}0\\2\\2\end{pmatrix} = \begin{pmatrix}0\\2t+2\\2t+2\\4t+4\end{pmatrix}, \quad T\begin{pmatrix}0\\-1\\0\end{pmatrix} = \begin{pmatrix}-1\\-t\\-1\\-t\end{pmatrix}.$$

Calculate the standard matrix of T. For what values of t is T one-to-one? For what value of t is T onto? Justify your answer.

$$\left(T\begin{pmatrix}1\\0\\1\end{pmatrix} - \tfrac{1}{2}T\begin{pmatrix}0\\2\\2\end{pmatrix}\right) - T\begin{pmatrix}0\\-1\\0\end{pmatrix} = T\begin{pmatrix}1\\-1\\0\end{pmatrix} - T\begin{pmatrix}0\\-1\\0\end{pmatrix} = T\begin{pmatrix}1\\0\\0\end{pmatrix} = \underbrace{\begin{pmatrix}0\\1\\t+1\\t+2\end{pmatrix} - \begin{pmatrix}0\\t+1\\t+1\\2t+2\end{pmatrix}}+\begin{pmatrix}1\\t\\1\\t\end{pmatrix} = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

$$-T\begin{pmatrix}0\\-1\\0\end{pmatrix} = T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\t\\1\\t\end{pmatrix}$$

$$\frac{1}{2}T\begin{pmatrix}0\\2\\2\end{pmatrix} + T\begin{pmatrix}0\\-1\\0\end{pmatrix} = T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\t+1\\t+1\\2t+2\end{pmatrix} - \begin{pmatrix}1\\t\\1\\t\end{pmatrix} = \begin{pmatrix}-1\\1\\t\\t+2\end{pmatrix}$$

Standard matrix: $\begin{pmatrix}1 & 1 & -1\\0 & t & 1\\1 & 1 & t\\0 & t & t+2\end{pmatrix}$

Is it one-to-one? $\Rightarrow \begin{pmatrix}1 & 1 & -1\\0 & t & 1\\1 & 1 & t\\0 & t & t+2\end{pmatrix} \to \begin{pmatrix}1 & 1 & -1\\0 & t & 1\\0 & 0 & t+1\\0 & t & t+2\end{pmatrix}$

$\downarrow$

$\begin{pmatrix}1 & 1 & -1\\0 & t & 1\\0 & 0 & t+1\\0 & 0 & 0\end{pmatrix}$

It's one to one when $t \neq -1$ and $t \neq 0$

This matrix cannot be onto because there cannot be a pivot in every row (it is not square)

3.4 [Alex]

4:37 PM Sat Mar 18 100%

Math example concept problem
Mar 17, 2023 at 3:38 PM

On Standard Matrix:

due to getting 2 problems wrong on quiz on same concept

Definition: Standard matrix for linear transformation T is $[T(\vec{e}_1)] \ldots [T(\vec{e}_n)]$ where $T: \underset{\text{input}}{\mathbb{R}^n} \to \underset{\text{output}}{\mathbb{R}^n}$

Example: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. What is the standard matrix representing T?

$$\vec{e}_2 \nearrow T\left(\begin{bmatrix} 0 \\ 4 \end{bmatrix}\right) = T\left(4\begin{bmatrix} 1 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = 4\begin{bmatrix} 3 \\ 5 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 16 \end{bmatrix}$$

$$\vec{e}_1 \nearrow T\left(\begin{bmatrix} 4 \\ 0 \end{bmatrix}\right) = T\left(-8\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 6\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = -8\begin{bmatrix} 3 \\ 5 \end{bmatrix} + 6\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -18 \\ -28 \end{bmatrix}$$

$$\text{standard matrix} = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix}$$

$$A = \begin{bmatrix} -\frac{9}{2} & \frac{5}{2} \\ -7 & 4 \end{bmatrix}$$

4 Matrix algebra (inverses, multiplication)

4.1 [Ardy]

From the practice midterm:

"2. Write the inverse of $\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 1 & 2 & 3 \end{bmatrix}$.

For the matrix to be invertible, $\det \neq 0$.

Solving for the determinant by reducing to row echelon form:

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow{-R_1 + R_2 \text{ and } R_3} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Can multiply the diagnol values of a triangular matrix to find the determinant.

$\det = (1)(1)(-1) = -1$ ← doesn't equal 0, so our matrix is invertible.

• $[A \mid I] = [I \mid A^{-1}]$ - solve by putting this into RREF

Inverse: $$\begin{bmatrix} 1 & 2 & 4 & 1 & 0 & 0 \\ 1 & 3 & 5 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -3 & 0 & 4 \\ 0 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{bmatrix} \rightarrow \begin{array}{c} \overbrace{\qquad\quad}^{I} \; \overbrace{\qquad\quad}^{A^{-1}} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{bmatrix} \end{array}$$

$$A^{-1} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

How to check if invertible matrix is correct:

$$[A^{-1} | I] = [I | A]$$

$$A^{-1} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Check: $\begin{bmatrix} 1 & -2 & 2 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$

$$\to \begin{bmatrix} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & -3 & 5 & 2 & 1 & 0 \\ 0 & 2 & -3 & -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{5}{3} & -\frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 2 & -3 & -1 & 0 & 1 \end{bmatrix}$$

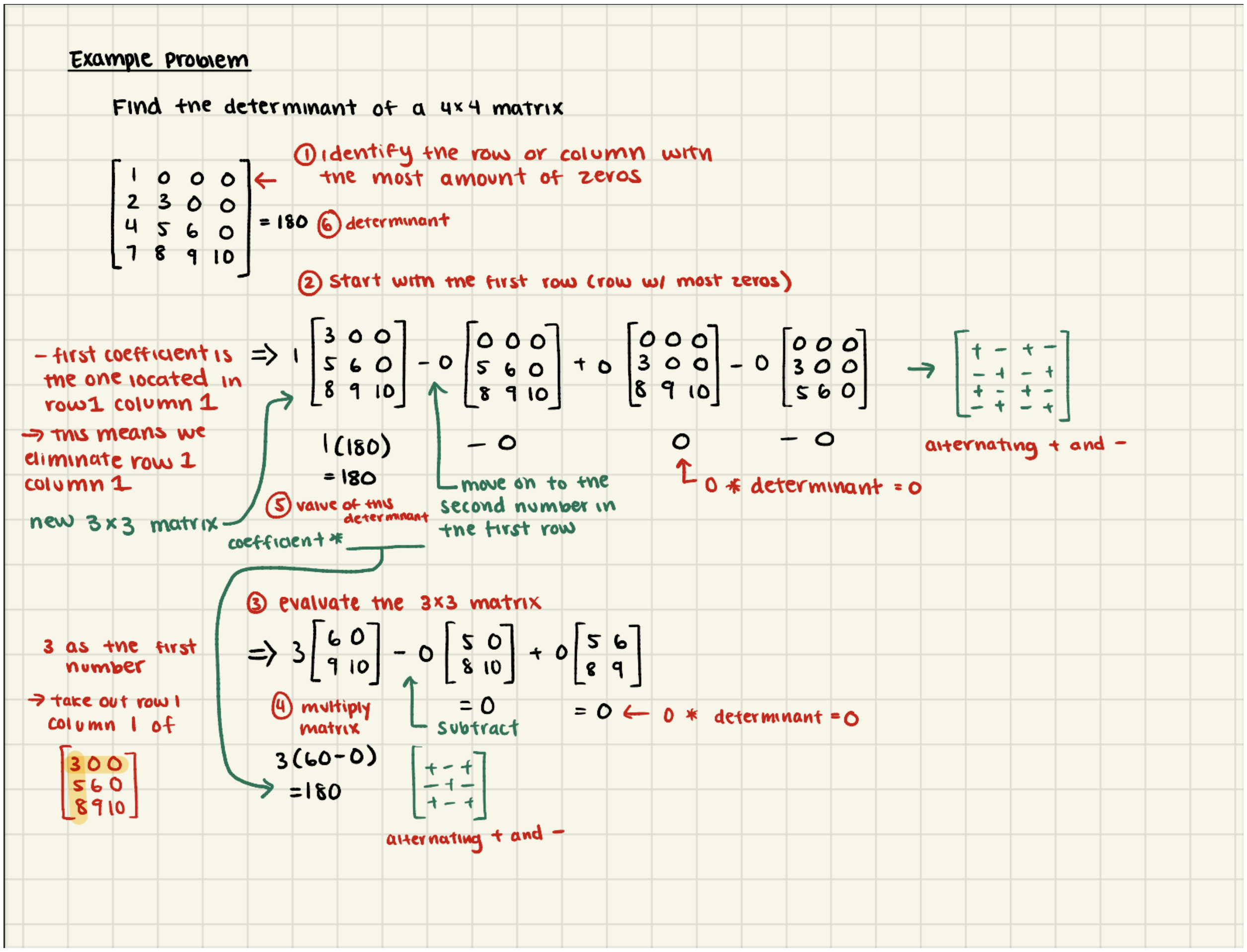
$$\to \begin{bmatrix} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{5}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{5}{3} & -\frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & 0 & -1 & -4 & -6 \\ 0 & 1 & 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 1 & 2 & 3 \end{bmatrix}$$

We got $A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 1 & 2 & 3 \end{bmatrix}$, which checks out.

5 Determinants

5.1 [Emily]



Example Problem

Find the determinant of a 4×4 matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 10 \end{bmatrix} = 180$$

① identify the row or column with the most amount of zeros

⑥ determinant

② start with the first row (row w/ most zeros)

- first coefficient is the one located in row 1 column 1

→ this means we eliminate row 1 column 1

new 3×3 matrix

$$\Rightarrow 1\begin{bmatrix} 3 & 0 & 0 \\ 5 & 6 & 0 \\ 8 & 9 & 10 \end{bmatrix} - 0\begin{bmatrix} 0 & 0 & 0 \\ 5 & 6 & 0 \\ 8 & 9 & 10 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 8 & 9 & 10 \end{bmatrix} - 0\begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

$1(180) = 180$ -0 0 -0

alternating + and −

⑤ value of this determinant

coefficient *

move on to the second number in the first row

0 * determinant = 0

③ evaluate the 3×3 matrix

3 as the first number

→ take out row 1 column 1 of

$$\begin{bmatrix} 3 & 0 & 0 \\ 5 & 6 & 0 \\ 8 & 9 & 10 \end{bmatrix}$$

$$\Rightarrow 3\begin{bmatrix} 6 & 0 \\ 9 & 10 \end{bmatrix} - 0\begin{bmatrix} 5 & 0 \\ 8 & 10 \end{bmatrix} + 0\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$$

④ multiply matrix

$3(60-0) = 180$

$= 0$ subtract

$= 0 \leftarrow$ 0 * determinant = 0

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

alternating + and −

6 Vector spaces/subspaces

6.1 [Jason]

Q: determine whether the following are subspaces of $\mathbb{R}^n$.

$$H = \left\{ \begin{bmatrix} a+2b \\ a-b \\ 3b \end{bmatrix}, a.b \in \mathbb{R} \right\}$$

Definition of a subspace:

- A subspace of a vector space V is a subset H of V that has three properties:

1) The zero vector of V is in H.
2) H is closed under addition. That is, for each u and v in in H, the sum u+v is in H.
3) H is closed under multiplication by scalars. That is, for each u in H and each scalar c, the vector cu is also in H.

$$H = \left\{ \begin{bmatrix} a+2b \\ a-b \\ 3b \end{bmatrix}, a.b \in \mathbb{R} \right\}$$

(1) Say both a and b are 0,

$$\begin{bmatrix} a+2b \\ a-b \\ 3b \end{bmatrix} \rightarrow \begin{bmatrix} 0+2\cdot 0 \\ 0-0 \\ 3\cdot 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow$$

This clearly shows that the zero vector is in H. Therefore, the first property is met.

(2) say $u = \begin{bmatrix} a_1+2b_1 \\ a_1-b_1 \\ 3b_1 \end{bmatrix}$ and $V = \begin{bmatrix} a_2+2b_2 \\ a_2-b_2 \\ 3b_2 \end{bmatrix}$; $u \in H$, $V \in H$

$$u+V = \begin{bmatrix} a_1+a_2+2b_1+2b_2 \\ a_1+a_2-b_1-b_2 \\ 3b_1+3b_2 \end{bmatrix} = \begin{bmatrix} (a_1+a_2)+2(b_1+b_2) \\ (a_1+a_2)-(b_1+b_2) \\ 3(b_1+b_2) \end{bmatrix}$$

u+V matches the form of u and v. Therefore, we are able to prove that for any $u, V \in H$, $u+V \in H$.

We then let $a = a_1+a_2$ and $b = b_1+b_2$

$$u+V \in H \leftarrow \begin{bmatrix} a+2b \\ a-b \\ 3b \end{bmatrix}$$

(3) say $u = \begin{bmatrix} a_1+2b_1 \\ a_1-b_1 \\ 3b_1 \end{bmatrix}$ and there is a scalar c ; $c \in \mathbb{R}$

$$Cu = \begin{bmatrix} c(a_1+2b_1) \\ c(a_1-b_1) \\ c(3b_1) \end{bmatrix} = \begin{bmatrix} ca_1+c2b_1 \\ ca_1-cb_1 \\ c3b_1 \end{bmatrix}$$

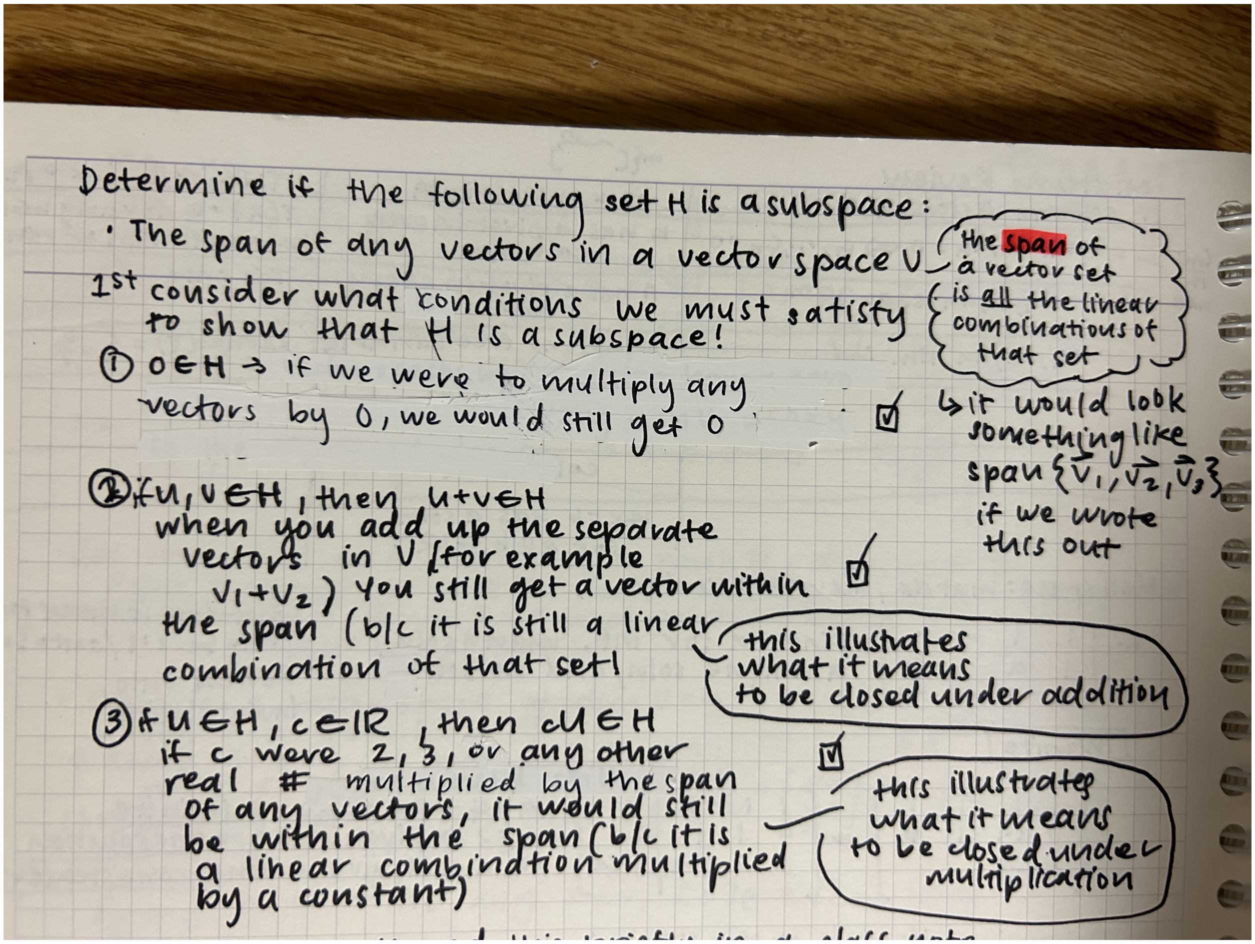
Cu matches the form of u. Therefore, we are able to prove that for any $c \in \mathbb{R}$ and $u \in H$, $Cu \in V$.

We then let $a = Ca_1$ and $b = Cb_1$

$$Cu \in H \leftarrow \begin{bmatrix} a+2b \\ a-b \\ 3b \end{bmatrix}$$

$\boxed{\text{Because all 3 properties are satisfied, we can show that H is a subspace.}}$

6.2 [Nikki]



Determine if the following set H is a subspace:

- The span of any vectors in a vector space V

The span of a vector set is all the linear combinations of that set

1^{st} consider what conditions we must satisfy to show that H is a subspace!

① $0 \in H$ → if we were to multiply any vectors by 0, we would still get 0 ☑

↳ it would look something like span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ if we wrote this out

② if $u, v \in H$, then $u+v \in H$
when you add up the separate vectors in V (for example $v_1 + v_2$) you still get a vector within the span (b/c it is still a linear combination of that set! ☑

this illustrates what it means to be closed under addition

③ if $u \in H$, $c \in \mathbb{R}$, then $cu \in H$
if c were 2, 3, or any other real # multiplied by the span of any vectors, it would still be within the span (b/c it is a linear combination multiplied by a constant) ☑

this illustrates what it means to be closed under multiplication

6.3 [Daniel R]

I will be working through the second practice problem from section 4.6 of the textbook:

Let H and K be subspaces of a vector space V. In section 4.1, Exercise 40, it is established that $H \cap K$ is also a subspace of V. Prove $\dim(H \cap K) \leq \dim H$.

- A subspace of V contains $\vec{0}$, is closed for addition, and is closed for scalar multiplication.

$H \cap K$ is a subspace of the vector space V. We know that for a finite-dimensional vector space V, $\dim H \leq \dim V$ if H is a subspace of V. Therefore, $\dim(H \cap K) \leq \dim H$, because $H \cap K$ is a subspace of V.

6.4 [Rebecca T]

306, 1405.

1. Let $\mathbb{P}_3$ be the space of all polynomials of degree at most 3.

Define H $\left\{ p(t) \in \mathbb{P}_3 \mid p(t) = at^3 + b, \text{ where } a, b \in \mathbb{R} \right\}$.

H is a subspace of $\mathbb{P}_3$?

1. $\boxed{0 \in H}$? prove $p(t) = 0$ due to $p(t) = at^3 + b$

$a, b \in \mathbb{R}$ $\quad a = b = 0 \in \mathbb{R}$

$\vec{0} \in \mathbb{R}$ ✓

2. $\boxed{\vec{u}, \vec{v} \in H \Rightarrow \vec{u} + \vec{v} \in H}$?

$u = a_1 t^3 + b_1 \quad v = a_2 t^3 + b_2 \Rightarrow u, v \in H$ ✓

$\vec{u} + \vec{v} = (a_1 t^3 + b_1) + (a_2 t^3 + b_2)$

$\vec{u} + \vec{v} \Rightarrow (a_1 + a_2) t^3 + (b_1 + b_2) \in H.$ ✓

$\vec{u} + \vec{v} \in H$ because $a_1 + a_2$ & $b_1 + b_2$ are the constant and also in the elements of $\mathbb{R}$. Therefore cont. plus cont. will get another constant.

3. $\boxed{\vec{u} \in H, \ c\vec{u} \in \mathbb{R}}$? $\qquad y = \overset{\text{coefficient}}{a}x + b \leftarrow \text{constant}$

$\vec{u} = a_1 t^3 + b_1 \qquad c \in \mathbb{R}$

$c\vec{u} = c(a_1 t^3 + b_1) \qquad c\vec{u} = ca_1 t^3 + cb_1$

$ca_1 \in \mathbb{R}$ & $cb_1 \in \mathbb{R}$.

$\therefore c\vec{u} \in \mathbb{R}$. ✓

6.5 [Daniel P]

Study Strategy

I like to organize the exam content by chapter so that I can get a full grasp of what the exam covers. Then I go over important concepts and theorems for each chapter. Then I solve problems for each chapter. I would review difficult questions. I would then solve the past exams to replicate the exam environment.

For the following subsets, determine if it is a subspace. If it is, compute its dimension.

(a) $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; 3x + y = z \right\} \subseteq \mathbb{R}^3$

① $0 \in R^3$

② $\underset{\vec{x}}{\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}} + \underset{\vec{y}}{\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}} = \begin{bmatrix} 2 \\ 3 \\ 9 \end{bmatrix}$ $\quad \vec{x}, \vec{y} \in \mathbb{R}^3$, $\quad \vec{x} + \vec{y} \in \mathbb{R}^3$

③ $\underset{\vec{x}}{\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}} \times \underset{c}{(-2)} = \begin{bmatrix} -2 \\ -2 \\ -8 \end{bmatrix}$ $\quad \vec{x} \in \mathbb{R}^3$, $\quad c\vec{x} \in \mathbb{R}^3$

subspace

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (z-y)/3 \\ y \\ z \end{bmatrix} \to \underbrace{y \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix}}_{\text{dimension} = 2}$$

Since $3x+y=z \rightarrow x=(z-y)/3$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (z-y)/3 \\ y \\ z \end{bmatrix} = y\underbrace{\begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}}_{v_1} + z\underbrace{\begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix}}_{v_2}$$

① v_1, v_2 are linearly ind. because $\begin{bmatrix} -1/3 & 1/3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow$ linearly independent.

② every element of $\mathbb{R}^3$ is a linear combo of v_1, v_2 since y and z can be any number.

$y\begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^3$. Therefore, $\mathbb{R}^3 = \text{span}\left\{\begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix}\right\}$

7 Matrix transformations (null spaces, column spaces)

7.1 [Ava]

Finding Basis of a Null Space

$$A = \begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

To find null space solve $A\vec{x} = \vec{0}$ to find the free variables

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Pivot columns are basic variables, write out linear equations from rows. Rearrange to show basic variables in terms of free variables

$x_1 - 2x_2 + 4x_4 = 0 \Rightarrow x_1 = 2x_2 - 4x_4$

$x_3 - 9x_4 = 0 \Rightarrow x_3 = 9x_4$

$x_5 = 0$

Take these equations and express them as a vector. Write free variables as themselves.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 - 4x_4 \\ x_2 \\ 9x_4 \\ x_4 \\ 0 \end{bmatrix}$$

Then factor out free variables: $x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix}$

This $x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix}$ forms a linear combination where x_2 and x_4 can be anything

(because they are free variables so can be any constant $\in \mathbb{R}$ and $A\vec{x} = \vec{0}$ will still be true)

Thus because they are a linearly independent set of vectors that include (AKA span) all possible solutions for $A\vec{x} = 0$ (AKA the Null space) these vectors form the basis (A)

$$\text{Basis}(A) = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix} \right\}$$

7.2 [Wesley]

Determine the dimensions of $Nul(A)$, $Col(A)$, and $Row(A)$ for the matrix below:

$$A = \begin{bmatrix} 1 & 0 & 9 & 5 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

$\dim(Row(A)) = 2$, because $Row(A)$ comprises of all nonzero rows.
$\dim(Col(A)) = 2$, since there are 2 columns that contain pivots.
$\dim(Null(A)) = 2$, by the Rank-Nullity Theorem, $n = \dim(Col(A)) + \dim(Null(A))$, and $n = 4$ in this case.

8 Bases

8.1 [Emma]

Given that $H = \text{span}\{v_1, v_2\}$ and $B = \{v_1, v_2\}$. Show x is in H and find B-coordinate vector of x for

$$v_1 = \begin{bmatrix} 11 \\ -5 \\ 10 \\ 7 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 14 \\ -8 \\ 13 \\ 10 \end{bmatrix}, \quad x = \begin{bmatrix} 19 \\ -13 \\ 18 \\ 15 \end{bmatrix}$$

$[x]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$, B-coordinate of $x \in V$ relative to $\underset{\text{basis}}{B} = \{b_1, b_2 \ldots b_n\}$ are the weights $c_1, c_2, \ldots c_n$ (if $x = c_1 b_1 + \ldots + c_n b_n$)

↑ want

know $\vec{x} = P_B [\vec{x}]_B$ so $P_B = \begin{bmatrix} 11 & 14 \\ -5 & -8 \\ 10 & 13 \\ 7 & 10 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 19 \\ -13 \\ 18 \\ 15 \end{bmatrix}$

$$\begin{bmatrix} 11 & 14 \\ -5 & -8 \\ 10 & 13 \\ 7 & 10 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 19 \\ -13 \\ 18 \\ 15 \end{bmatrix}$$

↓ looks like

$A\{\vec{x}\} = b$

try to find x you need row reduce echelon form

$$A = \begin{bmatrix} 11 & 14 & 19 \\ -5 & -8 & -13 \\ 10 & 13 & 18 \\ 7 & 10 & 15 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -5/3 \\ 0 & 1 & 8/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$c_1 = -5/3 \quad c_2 = 8/3$

$$[x]_B = \begin{bmatrix} -5/3 \\ 8/3 \end{bmatrix}$$

8.2 [Hyewon]

TEXTBOOK PROBLEM FROM SECTION 4.4

26. Given vectors $\mathbf{u}_1, \ldots, \mathbf{u}_p$, and $\mathbf{w}$ in V, show that $\mathbf{w}$ is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_p$ if and only if $[\,\mathbf{w}\,]_{\mathcal{B}}$ is a linear combination of the coordinate vectors $[\,\mathbf{u}_1\,]_{\mathcal{B}}, \ldots, [\,\mathbf{u}_p\,]_{\mathcal{B}}$.

THEOREM 8 FROM THE TEXTBOOK:
"the coordinate mapping $x \mapsto [x]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$."

DEFINITION OF ISOMORPHISM FROM THE TEXTBOOK:
"two spaces are indistinguishable as vector spaces. Every vector space calculation in V is accurately reproduced in W, and vice versa."

?

$$c_1 u_1 + \ldots + c_p u_p = w$$

$$[c_1 u_1 + \ldots + c_p u_p]_\beta = c_1 [u_1]_\beta + \ldots c_p [u_p]_\beta = [w]_\beta$$

Because the mapping $u \mapsto [u]_\beta$ is a linear transformation, w is a linear combination of $u_1, \ldots, u_p$ if $[w]_\beta$ is a linear combination of $[u_1]_\beta, \ldots, [u_p]_\beta$ (and vice versa).

8.3 [Michelle]

For $B = \{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \}$ is basis for V, find P_B and find x

for $[x]_B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

solution: $P_B = \begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix}$ because the basis is $\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \}$ and $P_B = [b_1 \; b_2]$

- $P_B [x]_B = x$ ← given in lecture
- $[x_B]$ is the vector of $c_0 \ldots c_n$ for ← by definition
- every vector x in V, where $x = c_1 b_1 + \ldots + c_n b_n$ ← because every vector in V is a linear combination of the basis B
  - ↳ $b_1 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ $b_2 = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$
- since $[x]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, $P_B = [b_1 \; b_2]$, and $P_B [x]_B = x$

$$\hookrightarrow \underbrace{\begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix}}_{P_B} \underbrace{\begin{bmatrix} 5 \\ 3 \end{bmatrix}}_{[x]_B} = \underbrace{\begin{bmatrix} 3 \\ -7 \end{bmatrix}}_{[x]}$$

$\boxed{\begin{bmatrix} 3 \\ -7 \end{bmatrix} \text{ is the solution}}$

9 Span

9.1 [Rebecca L]

Let $S = \{ v_1, \dots, v_p \}$ be a set in V, and let $H = \text{span}\{ v_1, \dots v_p \}$.

1. If one of the vectors in S, suppose v_b is a linear combination of the remaining vectors in S, then the set formed from S by removing v_b still spans H.

2. If $H \neq \{0\}$, some subset of S is a basis for H.

Example: Use the spanning set theorem to find a basis for $\text{Col}(A)$, where

$$A = [a_1, a_2 \dots a_5] = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 \\ 0 & 0 & 1 & 9 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $\underset{\text{column 2}}{a_2} = 4\underset{\text{row 1}}{a_1}$, $\underset{\text{column 4}}{a_4} = \underset{\text{row 1}}{-1a_1} + \underset{\text{row 2}}{9a_2}$, the Basic $= \{ a_1, a_3, a_5 \}$.

The pivot columns of a matrix A form a basis for $\text{Col}(A)$.

Example 2:

Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 7 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 2 \\ 11 \\ 9 \end{bmatrix}$

a. Show that $\text{Span}\{ v_1, v_2, v_3 \} = \text{Span}\{ v_1, v_2 \}$.

Let x be any vector in H

$$x = c_1 v_1 + c_2 v_2 + c_3 v_3$$

Since $v_3 = 2v_1 + v_2$, we substitute

$$x = c_1 v_1 + c_2 v_2 + c_3 (2v_1 + v_2)$$
$$= (c_1 + 2c_3) v_1 + (c_2 + c_3) v_2$$

This proves x is in $\text{Span}\{ v_1, v_2 \}$; every vector in H belongs to $\text{Span}\{ v_1, v_2 \}$.

b. Find a basis for the subspace H.

$\{ v_1, v_2 \}$ is the basic of H because they are linearly independent.