Math 53 Dis 108/111 Worksheet 7 (09/18)

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1 Chain Rule.

Problem 1. Let $z = f(x, y)$, $x = \sin(rs)$ and $y = rs$.

(a) Let $w(x, y)$ be any (nice/differentiable) function. Compute $\frac{\partial w}{\partial r}$.

(b) Using part (a), compute $\frac{\partial f_x}{\partial r}$ (let $w = f_x$).

(c) Use the product rule to compute $\frac{\partial}{\partial r}[w(x, y)g(r, s)]$ (where g is any nice/differentiable function).

(d) Use part (c) to compute $\frac{\partial}{\partial r}[f_x(x, y)g(r, s)]$.

(e) Finally, use everything you've learned in this problem to compute $\frac{\partial^2}{\partial r \partial s} z$.

2 Directional Derivative/Gradient.

Problem 2. Let $\vec{\mathbf{u}} \in \mathbb{R}^n$ be a nonzero vector. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function.

(a) Define (in plain English) the directional derivative of g in the direction $\vec{\mathbf{u}}$ at the point $p \in \mathbb{R}^n$.

(b) Using your definition from (a), define the directional derivative mathematically. Explain how the math corresponds to the English.

Problem 3. Consider the function

$$f(x, y) = \cos(x)e^y$$

Consider its derivative in the direction $\langle a, b\rangle$ at the point $(0, 0)$.

(a) Estimate the derivative in 3 different directions (i.e. do not compute any derivatives!).

(b) Give yourself a curve $(x(t), y(t))$ with tangent vector $\langle a, b\rangle$ at $(0, 0)$ and compute $\frac{d}{dt} f(x(t), y(t))$. (Hint: try using a line for your curve).

(c) Use the formula $D_{\langle a,b\rangle} f(0, 0) = \nabla f(0, 0) \cdot \langle a, b\rangle$.

Problem 4. Now consider the function

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & x \geqslant 0 \\ \frac{-xy}{\sqrt{x^2+y^2}} & x < 0 \end{cases}$$

(a) Check that this function is continuous.

(b) Estimate the derivative in the directions $\langle 1, 1\rangle, \langle 1, 0\rangle, \langle 0, 1\rangle$.

(c) Calculate the derivative in the direction $\langle 1, 1\rangle$ at the point $(0, 0)$ in the same two ways as you did in problem 2.

(d) Explain what happens (try plotting the function!).

Problem 5. Define the meaning of the gradient of f in plain English. The gradient is only well-defined for a differentiable function. Using your definition, explain why the function in problem 4 is not a nice (differentiable) function.

3 Tangent Planes.

Problem 6. Find the tangent plane to

$$z = 2x^2 + y^2 - 5y \qquad \text{at } (1, 2, -4)$$

three ways

(a) Writing down a linear approximation.

(b) Finding two curves that live in the surface going through the desired point but at different angles and using the tangent vectors to form a plane.

(c) By using the level set of a function.

Problem 7. Find the tangent plane to the unit sphere in $\mathbb{R}^3$ at the point $(0, 1, 0)$ using the three methods from problem 6. Can you make all of the methods work? Which is easiest? If one of the methods failed, why?

Problem 8. Find the tangent plane to the surface in $\mathbb{R}^3$ defined by the set of all points where $f(x) + g(y) = h(z)$ at the point (a, b, c). Assume that f, g, h are all nice (differentiable, but not necessarily invertible) functions.

Problem 9. Estimate $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$ by using a linear approximation at the point $(3, 2, 6)$.

4 Maxima and Minima

Problem 10. What is the definition of a critical point for a 1-variable function? Explain in plain English why critical points are possible maxima/minima.

Problem 11. Translate your 1-variable definition of a critical point to a 2-variable definition. Can you state your 2-variable definition using the gradient? What conditions does your function have to satisfy for your definition to work? (Look back to problems 4 and 5).

Problem 12. Where is the minimum of $f(x, y) = x^2 + y^2$? Show that the minimum occurs at a critical point according to your definition.