Math 53 Dis 108/111 Practice Final (B)

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Problem 1. (Hutchings) Let $f(x, y, z) = x + y + z$. Find the maxima and minima of f subject to the constraint $x^2 + y^2 + 2z^2 \leqslant 10$.

Problem 2. Find the equation of the plane containing the curve

$$x = \cos(2t) + t, \qquad y = \sin^2(t) + \frac{t^2}{2}, \qquad z = -t(1 + t).$$

Problem 3. (Tataru) Evaluate

$$\int_0^1 \int_0^1 \sin(\max(x^2, y^2))\,\mathrm{d}x\,\mathrm{d}y.$$

Problem 4. Compute the area of the weird ellipse $\frac{(x-y)^2}{4} + (x - 2y)^2 \leqslant 1$.

Problem 5. (Textbook) Evaluate

$$\int_{-2}^{2} \int_0^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2 + y^2 + z^2}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y.$$

Problem 6. (Textbook) Let C be the curve $x = \cos(t), y = \sin(t), z = \sin(t)$ for $0 \leqslant t \leqslant \pi/2$. Compute

$$\int_C 2xe^{2y}\,\mathrm{d}x + (2x^2e^{2y} + 2y\arctan(z))\,\mathrm{d}y + y^2\frac{1}{1 + z^2}\,\mathrm{d}z$$

Problem 7. (Tataru) Let S be the unit sphere in 3d, i.e. $x^2 + y^2 + z^2 = 1$. Evaluate

$$\iint_S z^{2018}\,\mathrm{d}S$$

Problem 8. (Zworski) Let S be the part of the surface $z = 25 - x^2 - y^2$ above $z = 16$, with the downwards orientation. Let $\mathbf{F} = \left\langle \frac{1}{48}yz, \frac{1}{27}x^2y, \cos(xyz)\right\rangle$. Using Stokes' theorem, compute

$$\iint_S \operatorname{curl}\mathbf{F}\cdot\,\mathrm{d}\mathbf{S}.$$

Problem 9. (Hutchings) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function such that

$$\frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial y}(x, y).$$

Suppose that $f(2, 3) = 6$. Compute $f(4, 1)$.

Problem 10. (Srivastava) Let $\mathbf{F} = \langle y^3z, x^3z, 1 + e^{x^2+y^2}\rangle$. Let S be the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane (with the upwards orientation). Use the divergence theorem and the filled-in piece of paraboloid (i.e. the region $z \geqslant 0$ and $z \leqslant 1 - x^2 - y^2$) to compute

$$\iint_S \mathbf{F}\cdot\,\mathrm{d}\mathbf{S}.$$

Problem 11. True/False.

(a) (Srivastava) If $\text{curl}(\mathbf{F}) = \nabla f$, (where $\mathbf{F}$ is some vector field) then f solves $\frac{\partial^2}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$.

(b) (Srivastava) The flux of $\mathbf{F} = \langle x, 0, 0 \rangle$ across the sphere of radius 1 centered at the origin is strictly less than the flux across the sphere of radius 2 centered at the origin. (Both spheres have the outward orientation)

(c) If a and b are orthogonal and c is parallel to a, then b and c are orthogonal.

(d) If $f(x, y)$ is differentiable with $\frac{\partial f}{\partial x} = -1$ and $\frac{\partial f}{\partial y} = 2$, then there is a direction $\mathbf{u}$ such that $D_{\mathbf{u}} f = \sqrt{6}$.

(e) Suppose all the level sets of $f(x, y)$ are parallel lines. Then the graph of the function is a plane.

(f)

$$\int_0^1 \int_0^1 f(x) f(y)\, \mathrm{d}x\, \mathrm{d}y = \left(\int_0^1 f(x)\, \mathrm{d}x \right)^2$$