Math 53 Dis 108/111 Practice Final (A)

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Problem 1. Let $f(x, y, z) = 4(x-1)^2 + 4(y+2)^2 + z^2$ and let S be the surface given by $f(x, y, z) = 4$. Show that no tangent plane contains the point $(1, -2, -1)$.

Problem 2. (Srivastava) Let $f(x, y) = x^3/3 + y^3/3 + 5x - y$. Find and classify the critical points of $g(x, y) = |\nabla f(x, y)|^2$.

Problem 3. (Tataru)

(a) Let $f(x, y) = g(r)$ where g is a twice-differentiable function and r is the radius in polar coordinates. Compute

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

in terms of x, y and g (and derivatives of g).

(b) Does $f(x, y) = \ln(x^2 + y^2)$ satisfy $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$?

Problem 4. (Hutchings) Calculate

$$\int_0^4 \int_{\sqrt{x}}^2 e^{y^3} \, \mathrm{d}y \, \mathrm{d}x.$$

Problem 5. (Tataru) Find the volume of the solid bounded from above by the sphere $x^2 + y^2 + z^2 = 2$ and from below by the paraboloid $z = x^2 + y^2$.

Problem 6. Compute the area of the weird ellipse $\frac{(x-y)^2}{4} + (x - 2y)^2 \leqslant 1$.

Problem 7. (Hutchings) Let S be a surface contained in the plane $z = x + y$. Let S have surface area 2023. Let $\mathbf{F} = \langle 3, 4, 5 \rangle$. Compute

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}.$$

Problem 8. (Tataru) Consider the curve of intersection between the surface $x^2 + y^2 - z^2 = 1$ and the surface $y = x^2$. Let C be the part of that curve between $(1, 1, -1)$ and $(1, 1, 1)$. Let

$$\mathbf{F} = \left\langle \frac{1}{1+y+z}, \frac{-x}{(1+y+z)^2}, 2z + \frac{-x}{(1+y+z)^2} \right\rangle.$$

Compute

$$\int_L \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

Problem 9. (Srivastava) Let $\mathbf{F} = \langle y^3 z, x^3 z, 1 + e^{x^2+y^2} \rangle$. Let S be the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane (with the upwards orientation). Use the divergence theorem and the filled-in piece of paraboloid (i.e. the region $z \geqslant 0$ and $z \leqslant 1 - x^2 - y^2$) to compute

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}.$$

Problem 10. (Srivastava) Let S be the part of the paraboloid $z = 4 - x^2 - y^2$ in the first octant ($x \geqslant 0, y \geqslant 0, z \geqslant 0$). Let C be the boundary curve of S (with counterclockwise orientation when viewed from above). Use Stokes' theorem to compute

$$\int_C \langle yz, -xz, 1 \rangle \cdot \mathrm{d}\mathbf{r}.$$

Problem 11. True/False.

(a) (Srivastava) If $f(x, y)$ is differentiable with $\frac{\partial f}{\partial x} = -1$ and $\frac{\partial f}{\partial y} = 2$, then there is a direction $\mathbf{u}$ such that $D_{\mathbf{u}} f = 0$.

(b) (Srivastava) Consider the level curve of a differentiable function $f(x, y) = k$. If P is a point such that the level curve intersects itself non-tangentially, then P is a critical point of f.

(c) (Srivastava) If $\operatorname{curl}(\mathbf{F}) = \nabla f$, (where $\mathbf{F}$ is some vector field) then f solves $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$.

(d) (Srivastava) If $\mathbf{F}$ and $\mathbf{G}$ are conservative vector fields, then $\mathbf{F} + \mathbf{G}$ is also a conservative vector field.

(e) (Textbook) If $\mathbf{F}$ and $\mathbf{G}$ are a vector fields and $\operatorname{div} \mathbf{F} = \operatorname{div} \mathbf{G}$ then $\mathbf{F} = \mathbf{G}$.

(f) Let S be a surface defined as the level set $f(x, y, z) = k$ of a differentiable function f. Let $\gamma(t)$ be a curve in S. Then $\gamma'(0)$ and $\nabla f(\gamma(0))$ are parallel.