Math 53 Dis 108/111 Worksheet 16 (11/29)

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1 Derivative Calculus

1.1 Problem Types

These are the types that I find most likely.

I. Something about the meaning of level sets (i.e. that the value of the function does not change along the level set).

II. Find the tangent plane to a surface (given by a level set), and use that tangent plane to verify a statement.

III. Find maxima and minima of a function given a constraint using Lagrange multipliers.

IV. Compute a geometric relation (e.g. angle) between planes, lines, etc.

These are some other questions that are possible.

I. Classify critical points of a function using the D-test.

II. Use the chain rule to compute derivatives of a function.

III. Use Clairaut's theorem (equality of mixed partials) to show that a function doesn't exist; or to make calculation of a derivative easy.

IV. Compute the area contained by a parametric curve, or a loop of a parametric curve.

V. Relate the directional derivative and the gradient in order to find directions where f increases the most, the least, not at all.

1.2 Practice Problems

Problem 1. (Hutchings) Let $f(x, y, z) = x + y + z$. Find the maxima and minima of f subject to the constraint $x^2 + y^2 + 2z^2 \leqslant 10$.

Problem 2. Let $f(x, y, z) = (x-1)^2 + (y+2)^2 + z^2$. Let (x_0, y_0, z_0) be a point on the level set $f(x, y, z) = 2$. What is the normal vector to the level set at that point?

Problem 3. (Srivastava) Let $f(x, y) = x^3/3 + y^3/3 + 5x - y$. Find and classify the critical points of $g(x, y) = |\nabla f(x, y)|^2$.

Problem 4. (Srivastava) Find the extreme values of $f(x, y) = e^{-xy}$ on the region $x^2 + 4y^2 \leqslant 1$.

Problem 5. Find the equation of the plane containing the curve

$$x = \cos(2t) + t, \qquad y = \sin^2(t) + \frac{t^2}{2}, \qquad z = -t(1 + t).$$

Problem 6. Let $f(x, y, z) = 4(x-1)^2 + 4(y+2)^2 + z^2$ and let S be the surface given by $f(x, y, z) = 4$. Show that no tangent plane contains the point $(1, -2, -1)$.

Problem 7. (Tataru)

(a) Let $f(x, y) = g(r)$ where g is a twice-differentiable function and r is the radius in polar coordinates. Compute

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

in terms of x, y and g (and derivatives of g).

(b) Does $f(x, y) = \ln(x^2 + y^2)$ satisfy $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$?

Problem 8. (Tataru) Consider the "asteroid" $x^2 + 2y^2 + 3z^2 \leqslant 1$.

(a) What is the equation of the tangent plane to the point (x_0, y_0, z_0) on the surface?

(b) What is the set of points on the surface that are visible from $(1, 1, 1)$? Describe the set using two equalities/inequalities.

Problem 9. (Tataru) Find the maximum and minimum of the function $f(x, y, z) = xyz + x + y + z$ on the filled-in sphere $x^2 + y^2 + z^2 \leqslant 1$.

Problem 10. True/False.

(a) (Srivastava) If $f(x, y)$ is differentiable with $\frac{\partial f}{\partial x} = -1$ and $\frac{\partial f}{\partial y} = 2$, then there is a direction $\mathbf{u}$ such that $D_{\mathbf{u}} f = 0$.

(b) If $f(x, y)$ is differentiable with $\frac{\partial f}{\partial x} = -1$ and $\frac{\partial f}{\partial y} = 2$, then there is a direction $\mathbf{u}$ such that $D_{\mathbf{u}} f = \sqrt{6}$.

(c) (Srivastava) Consider the level curve of a differentiable function $f(x, y) = k$. If P is a point such that the level curve intersects itself non-tangentially, then P is a critical point of f.

(d) Suppose all the level sets of $f(x, y)$ are parallel lines. Then the graph of the function is a plane.

(e) If a and b are orthogonal and c is parallel to a, then b and c are orthogonal.

2 Integral Calculus

2.1 Problem Types

I. Integral that is only doable by changing the order of integration.

II. Compute the volume of a 3d region, where the region is described by the boundary surfaces (below X, above Y, etc). This may require polar/cylindrical or spherical!

III. Change of variables.

2.2 Practice Problems

Problem 1. (Hutchings) Calculate

$$\int_0^4 \int_{\sqrt{x}}^2 e^{y^3} \, \mathrm{d}y \, \mathrm{d}x$$

Problem 2. (Hutchings) Let D be the region $x^2 + 3y^2 \leqslant 1$ calculate

$$\iint_D (x^2 + 3y^2)^{2023} \, \mathrm{d}A.$$

Problem 3. Compute the area of the weird ellipse $\frac{(x-y)^2}{4} + (x - 2y)^2 \leqslant 1$.

Problem 4. (Tataru) Find the volume of the solid bounded from above by the sphere $x^2 + y^2 + z^2 = 2$ and from below by the paraboloid $z = x^2 + y^2$.

Problem 5. (Tataru) Evaluate

$$\int_0^1 \int_0^1 \sin(\max(x^2, y^2))\, \mathrm{d}x\, \mathrm{d}y.$$

Problem 6. (Textbook) Evaluate

$$\int_{-2}^{2} \int_0^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2 \sqrt{x^2+y^2+z^2}\, \mathrm{d}z\, \mathrm{d}x\, \mathrm{d}y.$$

Problem 7. True/False.

(a)

3 Vector Calculus

3.1 Problem Types

These are the types that I find most likely.

I. Compute line integrals and line integrals of vector fields directly.

II. Recognize a conservative vector field, find the potential function, and use the fundamental theorem of line integrals.

III. Compute div, grad, curl, and use the identities div(curl) = 0 or div(grad) = 0 to simplify problems (together with some of the theorems below).

IV. Compute surface integrals and fluxes directly.

V. Use Stokes' theorem (usually to go from a surface to boundary integral)

VI. Use the divergence theorem (in 3d and/or in 2d) to evaluate an integral.

These are some other questions that are possible.

I. Prove an identity involving div, grad, or curl (e.g. $\nabla \cdot (u\nabla v) = \nabla u \cdot \nabla v + u \cdot (\mathrm{div}(\nabla v))$).

II. Prove that a vector field is/is not conservative.

III. Use Green's theorem.

3.2 Practice Problems

Problem 1. (Hutchings) Let $\mathbf{F} = \langle x^{2023} + y, y^{2023} + z, z^{2023} + x \rangle$. Let C the be curve that the intersection between the cylinder $x^2 + y^2 = 1$ and the plane $z = 2x + 3y$, with the counterclockwise orientation when viewed from above. Compute

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

Problem 2. (Textbook) Let C be the curve $x = \cos(t), y = \sin(t), z = \sin(t)$ for $0 \leqslant t \leqslant \pi/2$. Compute

$$\int_C 2xe^{2y}\, \mathrm{d}x + (2x^2e^{2y} + 2y \arctan(z))\, \mathrm{d}y + y^2 \frac{1}{1+z^2}\, \mathrm{d}z$$

Problem 3. (Hutchings) Let S be the upper hemisphere of the unit ball (i.e. $x^2 + y^2 + z^2 = 1$ and $z \geqslant 0$) with the upwards orientation. Let $\mathbf{F} = \langle x + \sin y, y + \cos z, z + 1\rangle$. Compute

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}.$$

Problem 4. (Hutchings) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable function such that

$$\frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial y}(x, y).$$

Suppose that $f(2, 3) = 6$. Compute $f(4, 1)$.

Problem 5. (Hutchings) Let S be a surface contained in the plane $z = x + y$. Let S have surface area 2023. Let $\mathbf{F} = \langle 3, 4, 5\rangle$. Compute

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}.$$

Problem 6. (Srivastava) Let S be the part of the paraboloid $z = 4 - x^2 - y^2$ in the first octant ($x \geqslant 0, y \geqslant 0, z \geqslant 0$). Let C be the boundary curve of S (with counterclockwise orientation when viewed from above). Use Stokes' theorem to compute

$$\int_C \langle yz, -xz, 1\rangle \cdot \mathrm{d}\mathbf{r}.$$

Problem 7. (Srivastava) Let L be the line segment from $(1, 0, 0)$ to $(4, 1, 2)$. Compute

$$\int_L \langle z^2, x^2, y^2\rangle \cdot \mathrm{d}\mathbf{r}$$

Problem 8. (Srivastava) Let $\mathbf{F} = \langle y^3 z, x^3 z, 1 + e^{x^2+y^2}\rangle$. Let S be the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane (with the upwards orientation). Use the divergence theorem and the filled-in piece of paraboloid (i.e. the region $z \geqslant 0$ and $z \leqslant 1 - x^2 - y^2$) to compute

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}.$$

Problem 9. (Tataru) Consider the curve of intersection between the surface $x^2 + y^2 - z^2 = 1$ and the surface $y = x^2$. Let C be the part of that curve between $(1, 1, -1)$ and $(1, 1, 1)$. Let

$$\mathbf{F} = \left\langle \frac{1}{1 + y + z}, \frac{-x}{(1 + y + z)^2}, 2x + \frac{-x}{(1 + y + z)^2} \right\rangle.$$

Compute

$$\int_L \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

Problem 10. (Zworski) Let S be the part of the surface $z = 25 - x^2 - y^2$ above $z = 16$, with the downwards orientation. Let $\mathbf{F} = \left\langle \frac{1}{48} yz, \frac{1}{27} x^2 y, \cos(xyz)\right\rangle$. Using Stokes' theorem, compute

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S}.$$

Problem 11. (Tataru) Let S be the unit sphere in 3d, i.e. $x^2 + y^2 + z^2 = 1$. Evaluate

$$\iint_S z^{2018} \, \mathrm{d}S$$

Problem 12. (Zworski) Let S be the surface given by rotating the curve $x = \cos(t)$ and $z = \sin(2t)$ about the z-axis where $t \in [-\pi/2, \pi/2]$. Use the divergence theorem to evaluate the volume of the solid contained by S. [Hint: use the vector field $\mathbf{F} = \langle 0, 0, z \rangle$]

Problem 13. True/False.

(a) (Srivastava) If $\operatorname{curl}(\mathbf{F}) = \nabla f$, (where $\mathbf{F}$ is some vector field) then f solves $\frac{\partial^2}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$.

(b) (Srivastava) If $\mathbf{F}$ and $\mathbf{G}$ are conservative vector fields, then $\mathbf{F} + \mathbf{G}$ is also a conservative vector field.

(c) (Srivastava) The flux of $\mathbf{F} = \langle x, 0, 0 \rangle$ across the sphere of radius 1 centered at the origin is strictly less than the flux across the sphere of radius 2 centered at the origin. Both sphere have the outward orientation.