Math 53 Dis 108/111 Worksheet 14 (11/08)

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1 Basic Skills

I. Parametrize a 2-d surface, or possibly parametrize only a part of a 2-d surface.

II. Find tangent planes of parametrized 2-d surfaces.

III. Compute surface areas.

IV. Know the terms **surface integral**, **flux**, **orientation**, **oriented/orientation**, **normal**.

2 Problems Types

I. Set up and compute integrals of functions over 2-d surfaces.

II. Set up and compute the flux of a vector field across a surface.

3 Practice

Problem 0 (Warm-up). Parametrize the following

(a) The unit circle centered at the origin.

(b) The circle of radius 2 centered at $(0,4)$.

(c) The ellipse centered at the origin with foci on the x-axis, and x-intercepts at ± 2, and y-intercepts at ± 1.

(d) The top half of the hyperboloid $y^2 = 1 + x^2$.

Problem 1. Parametrize the following

(a) The plane defined by the normal $\langle 1,1,1\rangle$ and the point $(1,0,0)$.

(b) The sphere of radius 1.

(c) The sphere of radius 2 centered at $(1,1,1)$.

(d) The ellipsoid $\frac{x^2}{4} + \frac{y^2}{2} + z^2 = 1$.

(e) The hyperboloid $y^2 + z^2 = 1 + x^2$.

Problem 2. Parametrize everything from **problem 1**, but with the additional restrictions.

(a) The part where $x > 0, y > 0, z > 0$.

(b) Inside the top half of the cone $z^2 = x^2 + y^2$.

(c) Where $x > 0$ (hint: think which choice of spherical coordinates will make this easiest)

(d) Where $y > 0$. Where $y > 1/4$ (hint: think which parametrization is easiest).

(e) Above the xy plane.

For all problems that follow, let S be a 2-d surface in $\mathbb{R}^3$ parametrized by $\mathbf{r}(u,v)$, where the surface is covered once as u, v range through D.

Whenever $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ are not colinear, the tangent plane at a point $(x_0, y_0, z_0) = \mathbf{r}(u_0, v_0)$ is defined by the point (x_0, y_0, z_0) and the normal vector $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}(u_0, v_0)$.

Problem 3. Find the equation of the tangent plane for the following.

(a) (Textbook 16.6.33) The surface $x = u + v$, $y = 3u^2$, $z = u - v$ at the point $(2, 3, 0)$.

(b) The unit sphere at the point $(0, 0, 1)$ using three ways: (1) draw and guess, (2) using the level set method, (3) by parametrizing the sphere (be careful how you parametrize it!)

(c) The hyperboloid from problem 1 at the point $(1, 1, 1)$

The formula for surface area of the parametrized surface S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, \mathrm{d}A.$$

Problem 4. Compute the areas of the following.

(a) The part of the plane $x + y + z = 1$ where $x > 0, y > 0, z > 0$ using two ways: (1) use geometry and (2) use the parametric formula for the surface area.

(b) The hyperboloid $y^2 + z^2 = 1 + x^2$ where $-2 \leqslant x \leqslant 2$.

(c) (Textbook 16.6.47) The part of the paraboloid $y = x^2 + z^2$ that lies within the cylinder $x^2 + z^2 = 16$.

(d) (Textbook 16.6.48) The spiral ramp given by $\mathbf{r}(u,v) = \langle u \cos v, u \sin v, v \rangle$ for $0 \leqslant v \leqslant \pi$ and $0 \leqslant u \leqslant 1$.

The integral of a scalar function f over the parametrized surface S is

$$\iint_S f(x,y,z)\,\mathrm{d}S = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,\mathrm{d}A.$$

Problem 5. Compute the following integrals.

(a) (Textbook example 3) $\iint_S z\,\mathrm{d}S$ where S is the part of the cylinder $x^2+y^2=1$ together with the bottom $x^1+y^2 \leqslant 1$ where $z=0$ and the top, which is the part of the plane $z=1+x$ that is directly above the bottom.

(b) (Textbook 16.7.6) $\iint_S xyz\,\mathrm{d}S$ where S is the cone parametrized as $x=u\cos v$, $y=u\sin v$, $z=u$ for $0 \leqslant v \leqslant \pi/2$ and $0 \leqslant u \leqslant 1$.

(c) (Textbook 16.7.14) $\iint_S y^2x^2$ where S is the part of the cone $y^2=x^2+z^2$ where $0 \leqslant y \leqslant 5$.

Assuming that S is orientable and that $\mathbf{v}=\mathbf{r}_u \times \mathbf{r}_v$ defines an orientation, and that $\mathbf{F}$ is a vector field over $\mathbb{R}^3$, the surface integral of $\mathbf{F}$ over S (aka the flux of $\mathbf{F}$ across S) is

$$\iint_S \mathbf{F}\cdot\,\mathrm{d}\mathbf{S} = \iint_S \mathbf{F}\cdot\frac{\mathbf{v}}{|\mathbf{v}|}\,\mathrm{d}S$$

Problem 6 Compute the fluxes of the following vector fields.

(a) (Textbook example 4) $\mathbf{F}=\langle y,x,z\rangle$ across the unit sphere.

(b) (Textbook 16.7.22) $\mathbf{F}=\langle z,y,x\rangle$ across the spiral ramp from **problem 4** with the upwards pointing normal.

(c) (Textbook 16.7.24) $\mathbf{F}=\langle -x,-y,z^3\rangle$ across the cone $z^2=x^2+y^2$ where $1 \leqslant z \leqslant 3$ with the downwards orientation.