Math 53 Dis 108/111 Worksheet 13 (11/06)

Tom Schang

1 Basic Skills

I. Compute divergence of vector fields.

II. Compute curl of vector fields.

III. Know the terms **conservative**, **incompressbible**, and **irrotational**.

IV. Know the **Laplace operator**.

2 Problem Types

I. Given a vector field $\mathbf{F}$ on $\mathbb{R}^3$, prove that it is or is not conservative.

II. Given a vector field $\mathbf{F}$, prove that it is or is not the curl of some other vector field.

III. Use the second vector form of Green's Theorem, i.e.

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}s = \iint_D \text{div } \mathbf{F} \, \mathrm{d}A$$

3 Divergence and Curl

Problem 0 (Warm-up).

1. Define "**curl**": what is the input space? what is the output space? for an item in the input space, how is the operation defined?
2. Define "**divergence**": what is the input space? what is the output space? for an item in the input space, how is the operation defined?

Problem 1. (Textbook 16.5.12) Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ and $f : \mathbb{R}^m \to \mathbb{R}$. State whether each expression is meaningful and explain why. Note that your answer may depend on n and m. If it does, you should be clear about for which m and/or n it makes sense and for which it does not.

(a) curl f

(b) grad $\mathbf{F}$

(c) div(grad f)

(d) grad(div $\mathbf{F}$)

(e) curl(curl $\mathbf{F}$)

(f) div(div $\mathbf{F}$)

(g) (grad f) $\times$ (div $\mathbf{F}$)

Problem 2. (Textbook 16.5.1-8) Compute the curl and the divergence of the following vector fields.

(a) $\langle xy^2z^2, x^2yz^2, x^2y^2z \rangle$

(b) $\langle \ln(2y+3z), \ln(x+3z), \ln(x+2y) \rangle$

Problem 3. (Textbook 16.5.19) Is there a vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that curl $\mathbf{G} = \langle x \sin y, \cos y, z - xy \rangle$?

Problem 4. (Textbook 16.5.23-29) Prove the following identities.

(a) div($\mathbf{F} + \mathbf{G}$)=div $\mathbf{F}$+ div $\mathbf{G}$.

(b) div($f\mathbf{F}$)= $f$div $\mathbf{F}$+$\mathbf{F} \cdot \nabla f$.

(c) div($\mathbf{F} \times \mathbf{G}$) = $\mathbf{G}\cdot$ curl $\mathbf{F}$ $-\mathbf{F}\cdot$ curl $\mathbf{G}$.

Problem 5. (Textbook 16.5.33) Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}$ be a vector field, D a simply connected, bounded domain with boundary C.

(a) Using Green's theorem, prove that

$$\iint_D \nabla \cdot \mathbf{F} \, \mathrm{d}A = \oint_C \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}s$$

where $\mathbf{n}$ is the outwards pointing normal (i.e. take the tangent vector to C at each point, normalize, and rotate by 90º clockwise)

(b) Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ and let $\nabla^2 g$ be defined as $\nabla \cdot (\nabla g)$. Prove

$$\iint_D f \nabla^2 g \, \mathrm{d}A = \oint_C f(\nabla g) \cdot \mathbf{n} \, \mathrm{d}s - \iint_D \nabla f \cdot \nabla g \, \mathrm{d}A.$$

Hint: use 5 (a) with a good choice of $\mathbf{F}$. Then, also use 4 (b).

(c) Prove the formula for integration by parts in 1-variable calculus, i.e.

$$\int_{[a,b]} f g' \, \mathrm{d}x = fg\Big|_a^b - \int_{[a,b]} f' g \, \mathrm{d}x.$$

Compare your proofs for (b) and (c).

4 Green's Theorem: Other Form

Problem 1. Suppose $\mathbf{G} = \text{curl} \left\langle x^2 y + \sin(e^x), x^2 + y^2 + z^2, \int_0^1 (x+s)(y-s)(z+s) \, \mathrm{d}s \right\rangle$. Compute

$$\oint_C \mathbf{G} \cdot \mathbf{n} \, \mathrm{d}s.$$

Problem 2. Let

$$f(x, y) = \begin{cases} \cos(\pi(x^2 + y^2)^2) & x^2 + y^2 \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

and let $D = \{x^2 + y^2 \leqslant 4\}$.

$$\iint_D \text{div } \nabla f \, \mathrm{d}A$$